Editorial

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Editorial

If the following remarks needed a title, it could be "A Quick Guide to Asset Pricing – or How Not to Win a Nobel Prize", "Who Needs Formulas?", or simply "Trees R Us". Luckily, an Editorial does not need a title, and it is not filtered through a severe refereeing process. The following is an attempt to clarify some concepts of mathematical finance by means of an example. In particular, we shall explain in what sense the price of a security is the expectation of the discounted payments.

The development of the "world" is modeled by a tree, which represents the relevant possibilities. The tree starts at time 0. At times t = 1,2,...,T the tree has progressively more branches, reflecting the arrival of additional information. In our example, we consider a model with t = 0,1,2 only. This does not mean that the world comes to an end at time 2. It means that we consider contingent payments at times 1 and 2 only.

The tree of our example has nine nodes (three at time 1 and six at time 2). At time 0 there is a market for securities. A security provides one or several contingent payments in the future; that is the payment at time t is a function of the node a time t. If we admit securities with possibly negative payments, the set of all securities is a vector space (of dimension 9 in the example). Figure 1 displays a security that consists of four positive contingent payments.

It is assumed that the market for the securities is a market with perfect competition. This means essentially that there is an equilibrium price for all securities, and that this price is a linear functional of the securities. A distinction has to be made between the equilibrium price and the actual price: the equilibrium price can be applied only to price "small" securities, that is securities whose trading does not disturb the equilibrium of the market.

For each node at times t = 1,...,T define a security which pays \$ 1 at that time, provided that the world passes through that node, and \$ 0 at all other nodes. The simple nature of these Kronecker securities is upgraded by their noble name: they are called Arrow-Debreu securities. An arbitrary security is a linear combination of the Arrow-Debreu securities, and hence its price is the corresponding linear combination of the prices of the Arrow-Debreu securities. The equilibrium prices can be observed in the

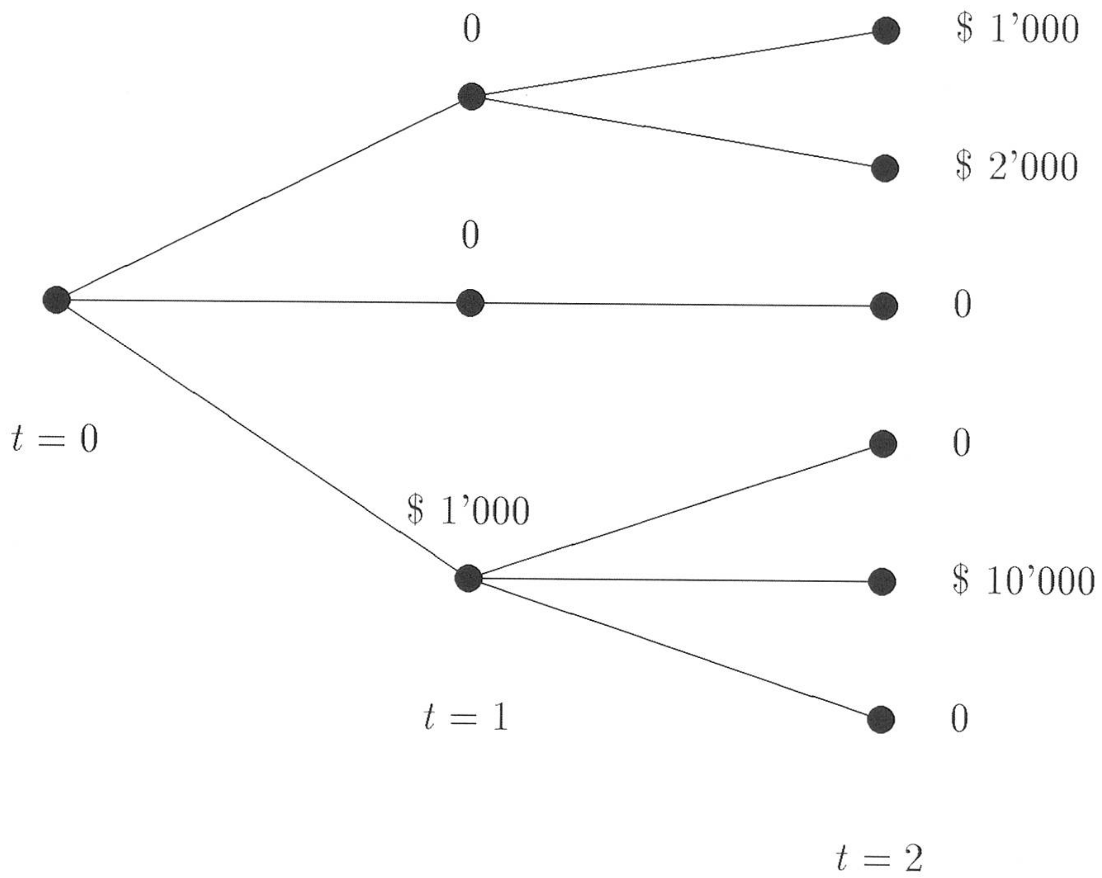

Figure 1: A Security with Four Contingent Payments

market, they are the basis of what follows. In our example, there are nine Arrow-Debreu securities; their prices are indicated in the respective node, see Figure 2. Thus the price of the security depicted in Figure 1 is

$$1000\,(.0576)+2000\,(.0864)+1000\,(.27)+10\,000\,(.0567) = \$\ 1067.40$$

So far we have considered the time-0 prices of the securities. Based on these, hypothetical future prices can be defined and calculated. Of course, these prices are conditional prices, because they are functions of the current node. In particular, we con-

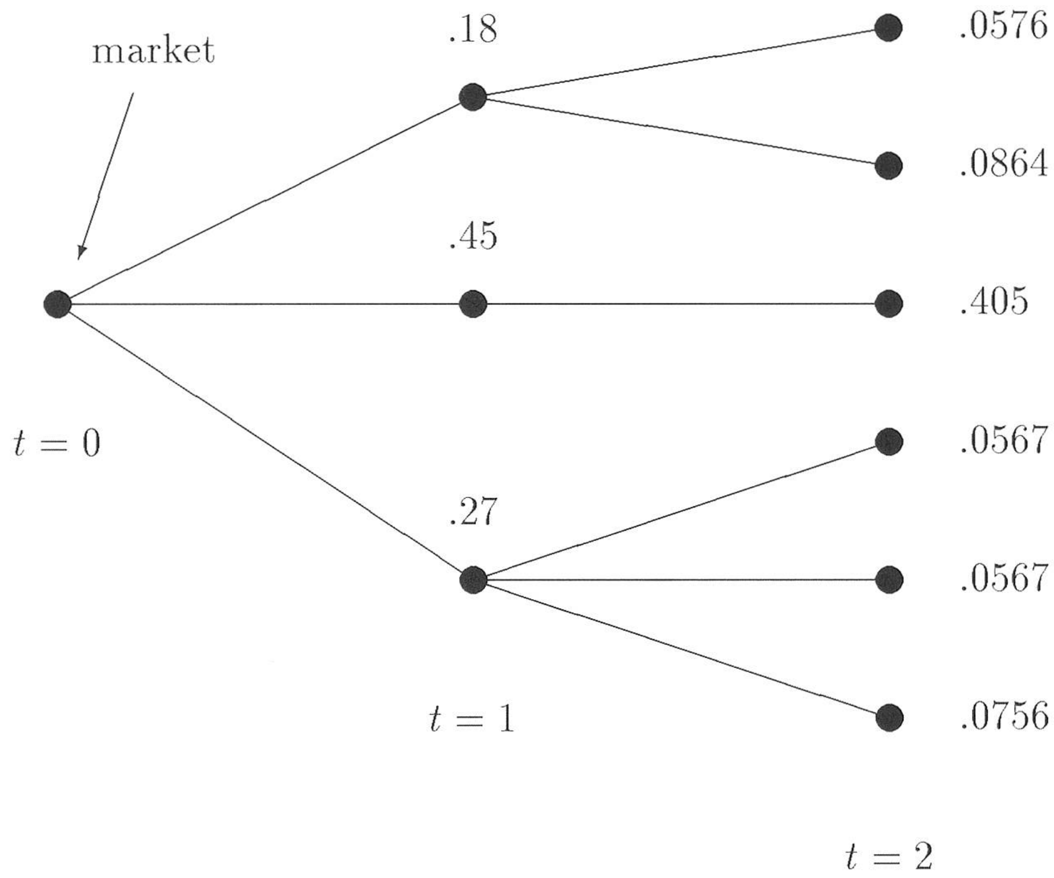

Figure 2: Time-0 Prices of the Arrow-Debreu Securities

sider the one period prices of the Arrow-Debreu securities. If we are in a given node at time t (t = 0,1,...,T-1), what is the price for receiving \$ 1 in a certain node at time t+1? The one period prices of the Arrow-Debreu securities are indicated in the respective branches of the tree, see Figure 3. They are obtained by the following reasoning: at time 0, the price of the Arrow-Debreu security with respect to a certain node at time t+1 must be the product of the price of the Arrow-Debreu security with respect to the preceding node at time t and the price for the corresponding one period Arrow-Debreu security.

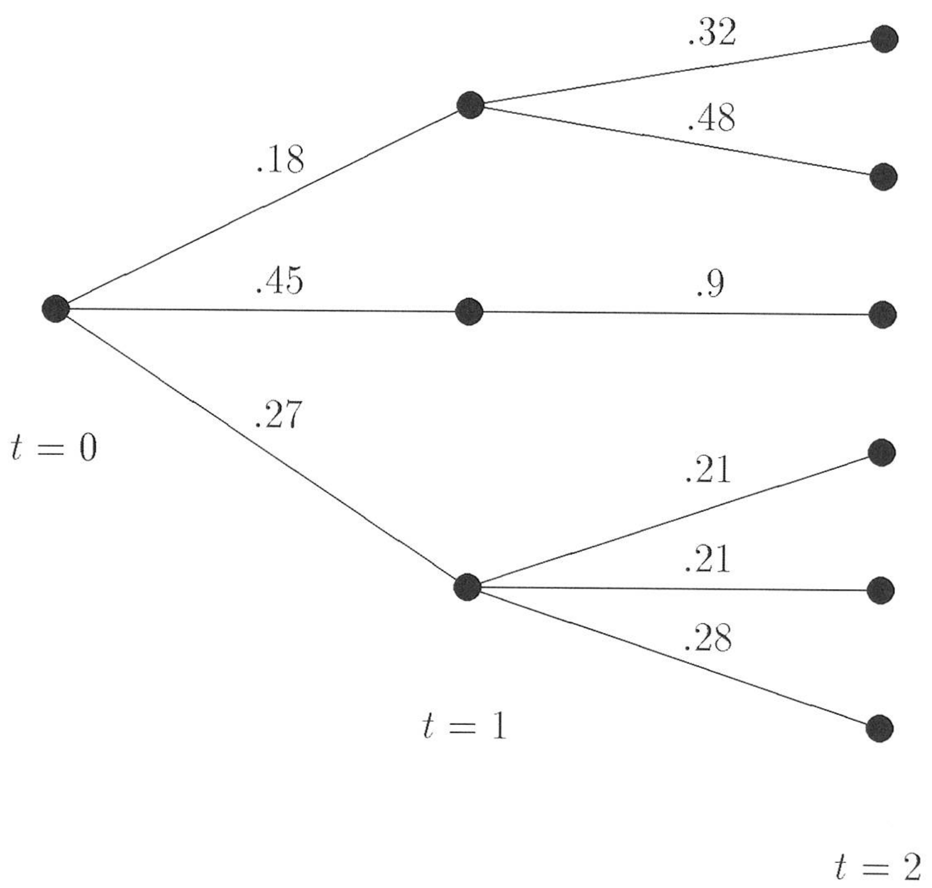

Figure 3: One Period Prices of the Arrow-Debreu Securities

Consider a node at a time t (t = 0,1,...T–1). Then it is possible to invest money in the market without any risk for one period at the risk-free interest rate. The corresponding one period risk-free discount factor is the price for receiving $ 1 with certainty at the end of the period. Naturally, this price depends on the current node, and it is obtained by summation of the corresponding one period prices of the Arrow-Debreu securities. The one period risk-free discount factors are indicated at the nodes, see Figure 4.

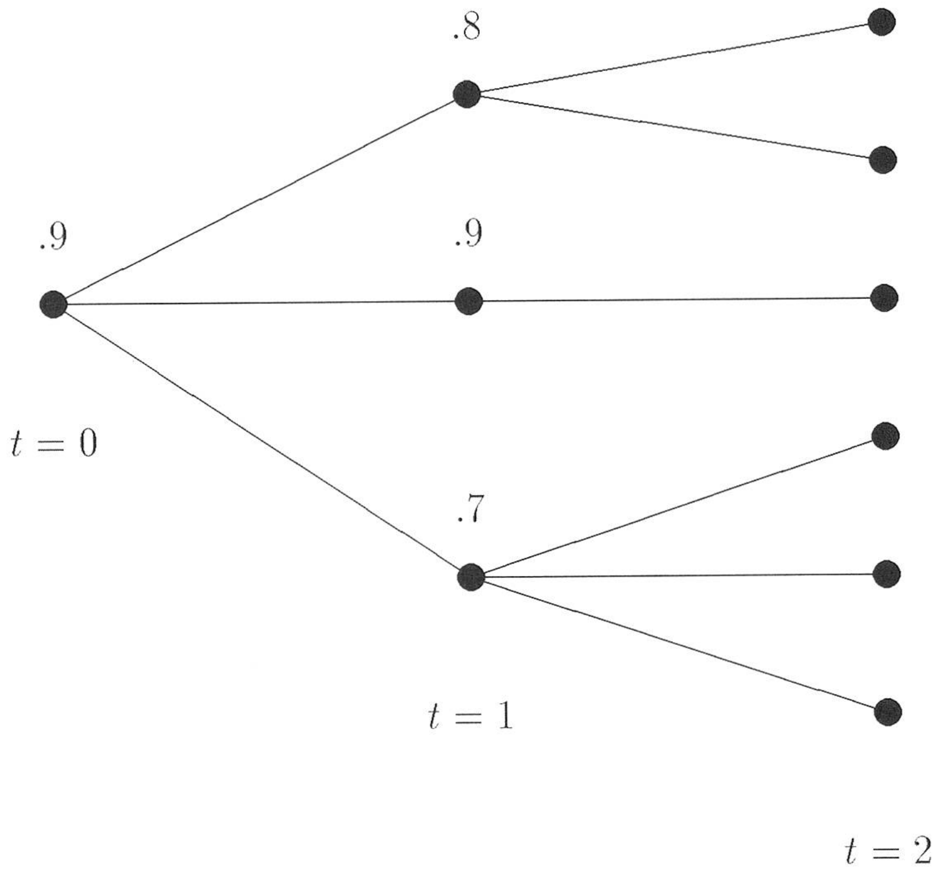

Figure 4: One Period Risk-free Discount Factors

The "risk-neutral" transition probabilities are a priori an artificial concept. For each branch, the transition probability is defined as a ratio: the one period price of the Arrow-Debreu security (Figure 3) divided by the one period risk-free discount factor at the preceding node (Figure 4). The results are shown in Figure 5. Note that the term transition probability is justified by the algebraic properties of these numbers. It does not mean that the world evolves from node to node according to these probabilities. In fact, we did not assume any probabilities at all for the tree. The only assumption concerned the prices.

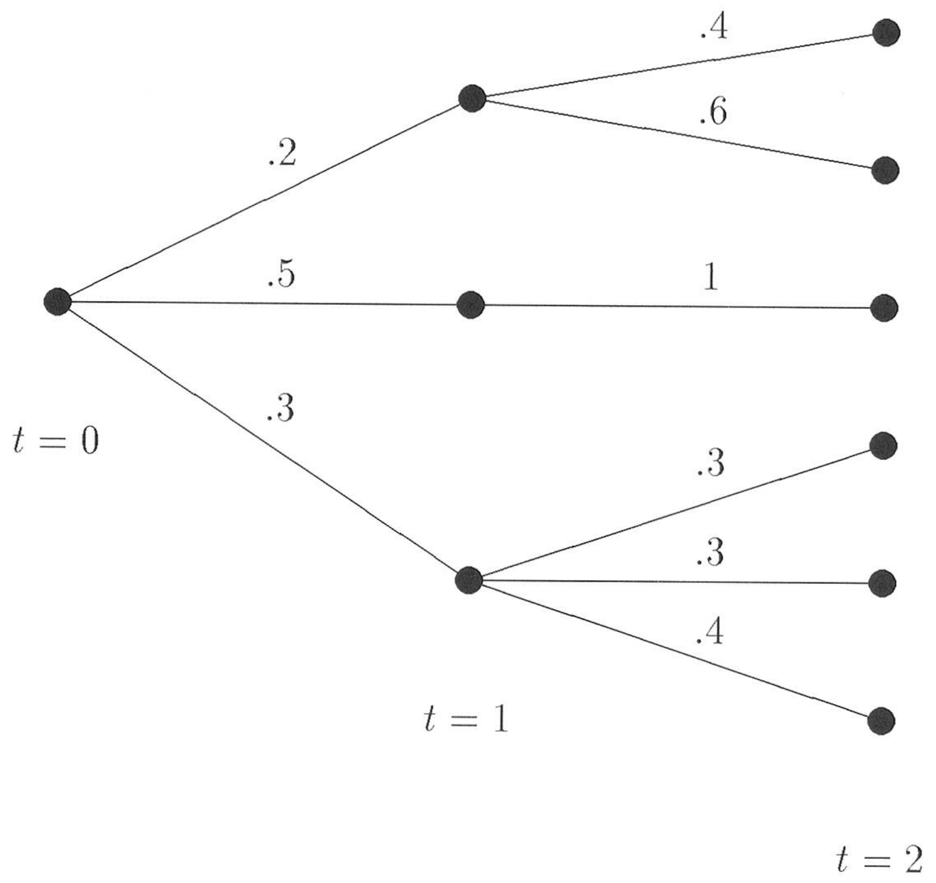

Figure 5: "Risk-neutral" Transition Probabilities

With these definitions, the price of a security becomes the sum of the expected discounted payments. Discounting a payment that is contingent to a certain node means multiplying the preceding one period discount factors, and calculating the expected value amounts to multiplying the preceding transition probabilities. For convenience, we have combined Figures 4 and 5 in Figure 6. Thus the price for the security of Figure 1 can be calculated as follows:

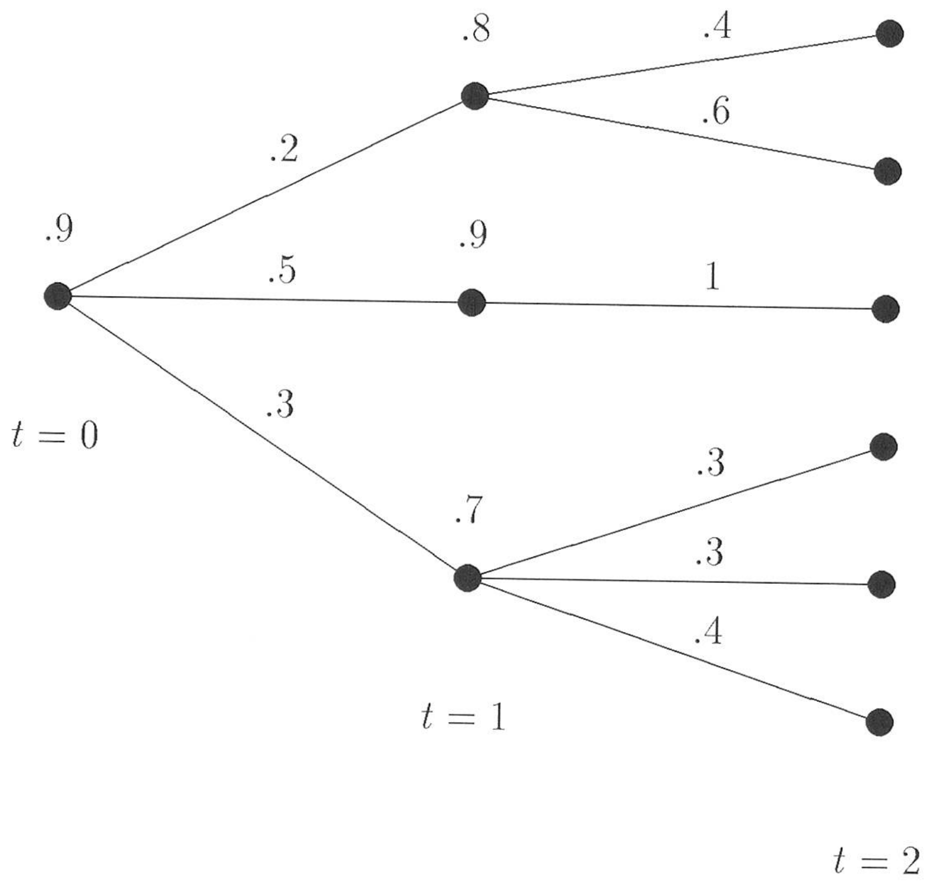

Figure 6: The Gist

$$(.9)\,(.2)\,(.8)\,(.4)\,1000 + (.9)\,(.2)\,(.8)\,(.6)\,2000 + (.9)\,(.3)\,1000 \\ + (.9)\,(.3)\,(.7)\,(.3)\,10\,000 = \$\,1067.40$$

To summarize: based on the observable time-0 prices of the securities, it is possible to define stochastic interest rates and transition probabilities on the tree, such that the price of any security can be written as the sum of the expected discounted payments.

Hans U. Gerber