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A Practical Application of Continuous Time Finance: Calculation of Benchmark Portfolios*

1 Introduction

1. In the modern asset allocation process the determination of a benchmark portfolio is of considerable importance. In principle one could define a dynamic benchmark strategy and measure the performance of the active portfolio relative to it. For practical and institutional reasons however, one prefers a constant portfolio as a benchmark. Typically this benchmark portfolio is calculated either with the one period Markowitz approach or with a special continuous time model. This leads to a constant investment strategy.
For the following reasons most practitioners currently use the one period Markowitz approach:

- The model is simple and easy to understand.
- Standard software for the Markowitz approach is commercially available.
- Many practitioners are interested in shortfall probabilities. Without much additional complexity shortfall constraints can be imposed in Markowitz optimization.

However, it is hard to find a foundation based on choice theory for the one period Markowitz approach without assuming asset returns to be multivariate normally distributed. This theoretical issue has rather unpleasant practical consequences:

- Given the chosen portfolio, one is typically interested in the distribution of wealth several years ahead. This leads to a product of normally distributed random variables. Quantiles and other characteristics of the distribution can only be calculated by simulation.
- Moreover, such a portfolio model is incompatible with International Capital Asset Pricing Models and the Black/Scholes Option Pricing Model, which are based on lognormal distributions (geometric Brownian motions).

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In this paper we favor the continuous time approach (Merton, 1990) for the following reasons:

- Given standard assumptions (geometric Brownian motion for asset prices, constant relative risk aversion) it can be shown that even under some types of constraints the optimal investment strategy is given by a portfolio which is constant over time (Cvitanic/Karatzas, 1992, Müller, 2000). In this case the dynamic stochastic optimization can be reduced to a static optimization problem as simple as Markowitz optimization.
- Under the special assumptions just mentioned, the optimal wealth process follows a geometric Brownian motion. Hence future wealth is lognormally distributed at each point of time. All characteristics can be calculated analytically and no simulation is needed.
- There is full compatibility with the assumptions of International Capital Asset Pricing Models and the Black/Scholes Model.

Two additional differences between the one period Markowitz approach and the continuous time models should be mentioned:

- The continuous time model assumes continuous rebalancing of portfolio weights, whereas in the Markowitz model no rebalancing takes place within one period (typically a year). In reality portfolios are rebalanced monthly or quarterly.
- In the continuous time model there is no natural way to model shortfalls. If there is a minimum target for future wealth the optimal portfolio choice typically becomes wealth and time dependent.

2. Within the continuous time framework we discuss an Asset Only Model and an Asset Liability Model. This distinction is motivated by the discrete one period model, where the covariance matrix is typically estimated on the basis of monthly or quarterly data. Moreover, a money market rate (3 or 12 months) is used as the risk free rate. Hence the implicit investment horizon is very short and the resulting asset allocation is inappropriate for private investors with long investment horizons and in particular for pension funds. To overcome this difficulty asset liability models are used. For pension funds one applies the Markowitz method on the surplus (assets minus liabilities) in the next period. Very often the liabilities of a pension fund are approximated by a fixed income portfolio. Analogously, for private investors with a long investment horizon, the asset portfolio is optimized relative to a long term fixed income index (Sharpe/Tint (1990), Keel/Müller (1995)).
This method can be adapted to the continuous time case (Müller (1999), (2000)). For a pension fund the funding ratio at the end of the planning horizon is

optimized. Under standard assumptions (geometric Brownian motion for asset prices and the liability, constant relative risk aversion) this problem can again be reduced to simple static optimization. Analogously, for a private investor optimization is applied on the ratio of final wealth of assets to the value attained under a reference strategy (e. g. given by a long term fixed income index).

3. For the Asset Only Continuous Time Model, as inputs one needs the covariance matrix of assets, expected returns of assets and the risk aversion of the investor:

- The covariance matrix is estimated on the basis of historical logarithmic returns. Our model contains less than 20 asset classes. Therefore variances and covariances can be estimated in a straightforward way. For models with more than 20 investment opportunities factor models are needed (see Grinold/Kahn (1995)). For the straightforward estimation of the covariance matrix typically monthly data over the last 5 years are used. An alternative method is based on more data, which enter the estimation with weights exponentially decreasing for the more distant past.
- The expected returns are calculated with the reverse optimization method. In the continuous time case the reverse optimization can be applied in full consistency with the Fisher Black (1989) model on Uniform Currency Hedging.
- Since our optimization is based on utility functions with constant relative risk aversion, the preferences of the investor can be fully characterized by the relative risk aversion parameter.

For the Asset Liability Continuous Time Model one has to define the liability. Moreover, the covariance matrix and expected returns of the assets and the liability have to be calculated. The preferences of the investor can still be represented by the relative risk aversion parameter. The following aspects are of some importance:

- The liability price process may be defined rather generally (e. g. the consumer price index, which is not fully trackable by assets) or more specifically by an asset portfolio (e. g. long term fixed income index).
- The covariance matrix and the vector of expected returns must include the liability as well. Strictly speaking, the expected return and the variance of the liability are not needed for the optimization in our model, but for illustrations of the results.

4. The rest of the paper is organized as follows: In section 2 the Asset Only Model is presented. Section 3 deals with the Asset Liability Model. Section 4 contains some concluding remarks.

2 The Asset Only Continuous Time Model

2.1 Theoretical Aspects

There is a riskless investment opportunity $i = 0$ with a constant logarithmic rate of return r and N risky investment opportunities $i = 1, \dots, N$ with geometric Brownian motions as price processes. Hence the price processes $S_{0,t}, \dots, S_{N,t}$ (adjusted for dividends, coupon payments, etc.) are given by

$$\begin{aligned} \frac{dS_0}{S_0} &= r\,dt \\ \frac{dS_i}{S_i} &= \mu_i\,dt + \sigma_i\,dZ_i\,, \quad i = 1, \dots N \end{aligned} \tag{1}$$

Besides

$$E(dZ_i) = 0\,, \quad \mathrm{Var}(dZ_i) = dt\,, \qquad i = 1, \dots, N$$

the Wiener processes $Z_1(t), \dots, Z_N(t)$ satisfy

$$\mathrm{Cov}(\sigma_i\,dZ_i, \sigma_j\,dZ_j) = V_{ij}\,dt \tag{2}$$

where V_{ij}, $i, j = 1, \dots, N$ denotes the covariances of the logarithmic returns. The relative allocation of total wealth in $t \in [0, T]$ is given by the portfolio choice

$$\hat{\boldsymbol{x}}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \qquad \text{with } \sum_{h=0}^{N} x_h(t) = 1 \tag{3}$$

Equivalently an investment strategy is given by the subportfolio of risky assets

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} \quad \text{and} \quad x_0(t) = 1 - \sum_{h=1}^{N} x_h(t)\,. \tag{4}$$

In a model without consumption this leads to a wealth process

$$\frac{dW_t}{W_t} = [r + \boldsymbol{\pi}^T \cdot \boldsymbol{x}(t)]\,dt + \sum_{i=1}^{N} x_i(t)\sigma_i\,dZ_i \tag{5}$$

with

$$\boldsymbol{\pi}^T = (\mu_1 - r, \ldots, \mu_N - r)\,. \tag{6}$$

Given initial wealth W_0, the investor chooses an investment strategy $\boldsymbol{x}^*(t)$, $t \in [0, T]$ maximizing the expected utility of final wealth under a set of constraints on the portfolio weights. Examples will be discussed below.
Formally, the optimization problem is given by

$$\max_{x(t)\in\mathbb{R}^N,\, t\in[0,T]} E[u(W_T)] \qquad \text{subject to} \qquad \begin{aligned} A\boldsymbol{x}(t) &\geq \boldsymbol{b} \\ C\boldsymbol{x}(t) &= \boldsymbol{d}\,. \end{aligned} \tag{O}$$

As usual the relative risk aversion is assumed to be constant. A constant relative risk aversion $c > 0$ leads to a utility function of the type

$$\begin{aligned} u(w) &= (1-c)w^{1-c}\,, && \text{for } c > 0\,, c \neq 1 \\ u(w) &= \ln w\,, && \text{for } c = 1 \end{aligned} \tag{7}$$

Under (1) and (7) it is well known, that (O) leads to a constant proportion investment strategy (constant portfolio weights) provided there are no constraints on the portfolio choice. In Cvitanić/Karatzas (1992), pp. 802–803 and in Müller (2000) it is shown that this result still holds under the constraints imposed in (O). Knowing that we have only to deal with constant proportion investment strategies $\boldsymbol{x}$ allows us to reduce (O) to a static optimization problem as simple as a traditional Markowitz optimization (Müller (1999) or Seiler (2000)).
For a constant proportion investment strategy $\boldsymbol{x}$ equation (5) becomes

$$\frac{dW_t}{W_t} = \mu_x\, dt + \sigma_x\, dZ \tag{8}$$

with

$$\begin{aligned} &Z(t) \quad \text{a Wiener process} \\ &\mu_x = r + \boldsymbol{\pi}^T\boldsymbol{x} \\ &\sigma_x^2 = \boldsymbol{x}^T V\boldsymbol{x}\,, \quad V = (V_{ij})\,,\ i, j = 1, \ldots, N\,. \end{aligned}$$

According to (8) (see e. g. Hull (1999), pp. 221–222), final wealth W_T is given by the lognormal random variable

$$W_T = W_0 \cdot \exp\left\{\left(\mu_x - \frac{\sigma_x^2}{2}\right)T + \sigma_x[Z(T) - Z(0)]\right\}. \tag{9}$$

Obviously for $c \neq 1$

$$u(W_T) = (1-c)W_0^{1-c} \cdot \exp\left\{(1-c)\left(\mu_x - \frac{\sigma_x^2}{2}\right)T + (1-c)\sigma_x[Z(T) - Z(0)]\right\} \tag{10}$$

is again lognormally distributed. According to the formula for the expected value of lognormal distributions (see e. g. Hull (1999), p. 229) one obtains

$$E[u(W_T)] = (1-c)W_0^{1-c} \cdot \exp\left\{(1-c)\left(\mu_x - \frac{\sigma_x^2}{2}\right)T + (1-c)^2\frac{\sigma_x^2}{2} \cdot T\right\} \tag{11}$$

or

$$E[u(W_T)] = (1-c)W_0^{1-c}\exp\left\{(1-c)\left[\mu_x - \frac{c}{2}\sigma_x^2\right]T\right\}. \tag{12}$$

Maximizing (12) is equivalent to the maximization of

$$\mu_x - \frac{c}{2}\sigma_x^2 \tag{13}$$

or

$$\frac{2}{c}(\mu_x - r) - \sigma_x^2 . \tag{14}$$

Hence (O) can be reduced to the static optimization

$$\boxed{\begin{array}{ll} \max\limits_{x \in \mathbb{R}^N} \{2\tau\boldsymbol{\pi}^T\boldsymbol{x} - \boldsymbol{x}^T V\boldsymbol{x}\}, & \text{with } \tau = \dfrac{1}{c} \\ \text{subject to} & \begin{array}{l} A\boldsymbol{x} \geq \boldsymbol{b} \\ C\boldsymbol{x} = \boldsymbol{d}. \end{array} \end{array}} \tag{OS}$$

1) The reduction of (O) to (OS) holds as well for the case $c = 1$. The derivation is straightforward.
2) (OS) is fully analogous to the Markowitz one period optimization. π and V refer now to the geometric Brownian motions defining the asset price processes (see (1)).

3) Without constraints the optimal solution $\boldsymbol{x}^*$ of (OS) is given by

$$\tau\boldsymbol{\pi} = V\boldsymbol{x}^* \tag{15}$$

On the one hand this leads to the famous Merton formula

$$\boldsymbol{x}^* = \tau V^{-1}\boldsymbol{\pi} \tag{16}$$

and on the other hand the condition

$$\boldsymbol{\pi} = \frac{1}{\tau}V\boldsymbol{x}^* \tag{17}$$

can be used as a basis for the "reverse optimization". The "reverse optimization" technique is widely used by practitioners for the estimation of the risk premia $\boldsymbol{\pi}$ (see below).

2.2 *Practical Implementation*

For the practical application of (OS) one needs the covariance matrix V, the vector of risk premia $\boldsymbol{\pi}$ and the risk tolerance parameter τ.

a. Estimation of the Covariance Matrix

Typically a benchmark portfolio contains less than 20 asset classes represented by indices (e. g. CH cash, CH bonds, CH equities, EURO cash, EURO bonds, EURO equities, etc.). In this situation the covariance matrix V may be estimated without using a factor model. An estimation based on historical data is generally accepted by the profession. Due to changes in the economic environment the stationarity assumption for return processes is questionable. Therefore data from the far distant past should have at most a negligible influence on the estimation. Moreover, due to the lack of stochastic independence daily or weekly data should not be used in a straightforward way. One generally accepted method uses monthly data over the last five years. As an obvious disadvantage the covariance matrix estimated by this method may change considerably five years after a crash has occurred. Another method is based on a longer history of monthly data. To cope with the nonstationarity this more sophisticated method attributes exponentially decreasing weights to observations in the more distant past. For the estimation the logarithmic excess returns relative to the riskless domestic cash are used. The covariance matrix is estimated using monthly data from Dec. 1987 until Jan. 2001. The weights for the world market portfolio are taken from Dec. 29, 2000. The weights for the world bond market portfolio are from the Merrill Lynch Global

Government and Global High Grade Corporate Indices and the weights for the world equity market portfolio come from the MSCI World Index.

b. Risk Premia

The optimal portfolio is highly sensitive to changes in risk premia. Therefore a precise estimation of the vector of risk premia is most important. Since returns are nonstationary one cannot use data reaching in the far distant past for a direct estimation of the risk premia. With only a few data over the last few years however, the estimation is not reliable. This can be illustrated by the Japanese equity market. In the early nineties an estimation of the risk premium based on the performance in the eighties would have led to disastrous results.

The problem can be circumvented by the reverse optimization technique. Formula (17) relates the vector of risk premia $\boldsymbol{\pi}$ to the optimal portfolio $\boldsymbol{x}^*$ and the risk tolerance $\tau > 0$. If $\boldsymbol{x}^M$ denotes the portfolio chosen by a representative investor with risk tolerance τ^M, then in the absence of constraints his choice is consistent with risk premia given by the reverse optimization formula

$$\boldsymbol{\pi} = \frac{1}{\tau^M} V \boldsymbol{x}^M . \tag{17}$$

The main difficulty consists in finding a good proxy for $\boldsymbol{x}^M$. Many practitioners use the Fisher Black Model on "Universal Currency Hedging" (1989). For this model Adler/Solnik (1990) pointed out that $\boldsymbol{x}^M$ has to be the partly currency hedged world market portfolio, given by

$$\begin{aligned} \boldsymbol{x}^M &= w_E[\boldsymbol{x}^E - (1-\tau^M)\boldsymbol{x}^{E,C}] \\ &\quad + (1-w_E)[\boldsymbol{x}^B - (1-\tau^M)\boldsymbol{x}^{B,C}] \end{aligned} \tag{18}$$

with

$\boldsymbol{x}^E$	world market portfolio for equities
$\boldsymbol{x}^B$	world market portfolio for bonds
$\boldsymbol{x}^{E,C}$	currency hedging portfolio for $\boldsymbol{x}^E$ (i. e. $\boldsymbol{x}^E - \boldsymbol{x}^{E,C}$ is fully hedged in each foreign currency)
$\boldsymbol{x}^{B,C}$	currency hedging portfolio for $\boldsymbol{x}^B$
$w_E = \dfrac{W_E}{W_E + W_B}$	with W_E, W_B world equity, respectively bond market capitalization.

Table 1 shows the world market and currency hedging portfolios $\boldsymbol{x}^E$, $\boldsymbol{x}^B$, $\boldsymbol{x}^{E,C}$, $\boldsymbol{x}^{B,C}$ used for our subsequent calculations (portfolio weights are subject to rounding errors).

	$\boldsymbol{x}^E$	$\boldsymbol{x}^B$	$\boldsymbol{x}^{E,C}$	$\boldsymbol{x}^{B,C}$	$\boldsymbol{x}^M$	π	π^{loc}
B CHF	%	0.4%	%	%	0.3%	0.3%	0.3%
E CHF	2.9				1.2	5.0	5.0
C EURO			18.7	33.2	–23.0	0.5	
B EURO		33.2			19.9	1.0	0.6
E EURO	18.7				7.5	6.1	5.6
C USD			55.3	40.2	–38.8	1.9	
B USD		40.2			24.1	2.4	0.5
E USD	51.3				20.5	7.0	5.1
C GBP			9.7	5.2	–5.9	1.3	
B GBP		5.2			3.1	2.2	0.9
E GBP	9.7				3.9	5.6	4.3
C JPY			13.4	21.0	–15.1	1.9	
B JPY		21.0			12.6	2.2	0.2
E JPY	13.4				5.3	6.3	4.4
E EMMA*	4.1				1.6	8.1	6.3
C CHF			–100.0	–100.0	82.8		

$\tau^M = 0.16\,,\quad w^E = 0.4$

* For emerging market equities (E Emma) USD is used as local currency

Table 1: Market portfolio and risk premia

Now the reverse optimization formula becomes

$$\begin{aligned}\boldsymbol{\pi} = \frac{1}{\tau^M} V\{&w_E[\boldsymbol{x}^E - (1-\tau^M)\boldsymbol{x}^{E,C}] \\ &+ (1-w_E)[\boldsymbol{x}^B - (1-\tau^M)\boldsymbol{x}^{B,C}]\}\end{aligned} \tag{19}$$

The parameters τ^M and w_E cannot be directly observed in the market[1]. For this reason one can use τ^M and w_E to calibrate the model. In fact these parameters are chosen such that the average risk premia on equities, bonds and currencies become consistent with historical observations reaching in the far distant past.

[1] Whereas reasonable estimates for the world market capitalization of equities W_E are available, the capitalization of bonds W_B cannot be reliably estimated.

For our calculations we used $\tau^M = 0.160$ and $w_E = 0.40$. This leads to the market portfolio $\boldsymbol{x}^M$ (standardized such that the sum of equity and bond holdings is 100 %) and the risk premia $\boldsymbol{\pi}$ in CHF shown in Table 1. Our choice of τ^M and w_E guarantees that the average risk premium on equities

$$\sum_{i=1}^{M} x_i^E \pi_i^{\text{loc}}$$

is[2] equal to 5 %. This level is consistent with very long run historical observations.

c. *Estimation of the Risk Tolerance τ for an Individual Investor*

Given some idea about the composition of the desired portfolio $\boldsymbol{x}^*$ (total weights of equities, bonds and cash) one can get a rough estimate of the investors risk tolerance τ. Transforming formula (15) leads to

$$\tau = \frac{\boldsymbol{x}^{*T} V \boldsymbol{x}^*}{\boldsymbol{\pi}^T \boldsymbol{x}^*} \,. \tag{20}$$

d. *Calculation of Optimal Portfolios*

If we use the covariance matrix V and the risk premia $\boldsymbol{\pi}$ calculated in 2.2.a, 2.2.b and put $\tau = \tau^M$, then in the absence of constraints the market portfolio $\boldsymbol{x}^M$ (see Table 1) must be the optimal solution of (OS). This is shown in Table 2. For later reference Table 2 contains as well the growth optimum portfolio ($\tau = 1$) and the optimal portfolios for $\tau = 0.1$, 0.2, 0,3. Of course in accordance to (16) all these portfolios are proportional. For the riskfree rate we assumed $r = 4.33\,\%$ which corresponds to the historical average over the sample period.

[2] π^{loc} results from π by subtracting from all foreign assets the corresponding cash risk premia.

τ	0.1	0.16	0.2	0.3	1
B CHF	0.2%	0.3%	0.3%	0.5%	1.6%
E CHF	0.7	1.2	1.4	2.2	7.2
C EURO	−14.4	−23.0	−28.7	−43.1	−143.7
B EURO	12.4	19.9	24.9	37.3	124.5
E EURO	4.7	7.5	9.3	14.0	46.7
C USD	−24.2	−38.8	−48.5	−72.7	−242.4
B USD	15.0	24.1	30.1	45.1	150.5
E USD	12.8	20.5	25.6	38.4	128.0
C GBP	−3.7	−5.9	−7.3	−11.0	−36.6
B GBP	1.9	3.1	3.9	5.8	19.3
E GBP	2.4	3.9	4.9	7.3	24.3
C JPY	−9.4	−15.1	−18.8	−28.3	−94.2
B JPY	7.9	12.6	15.8	23.6	78.8
E JPY	3.3	5.3	6.7	10.0	33.4
E EMMA	1.0	1.6	2.0	3.1	10.2
C CHF*	89.3	82.8	78.5	67.8	−7.4
Total	100.0	100.0	100.0	100.0	100.0
Equities	25.0	40.0	49.9	74.9	249.7
Bonds	37.5	60.0	74.9	112.4	374.6
Net CH Cash*	37.6	0.0	−24.9	−87.3	−524.3
Total	100.0	100.0	100.0	100.0	100.0
Currency Hedging*	51.7	82.8	103.4	155.1	517.0
μ_x	5.9	6.9	7.5	9.1	20.2
σ_x	4.0	6.4	8.0	11.9	39.8

* Foreign currency hedging is implemented by future or forward contracts. These contracts can be decomposed in short postions in foreign money markets and long positions in the domestic money market.

- C CHF includes the currency hedging part of the domestic money market position.
- Net Domestic Cash denotes the position in domestic cash not including the currency hedging part.
- Currency Hedging Positions denote the corresponding future or forward contracts.

Table 2: Optimization without constraints

Next, in table 3 it is shown how the optimal portfolios change if currency hedging, short selling and borrowing is excluded (i. e. $x_i \geq 0$, $i = 1, \dots, N$, $\sum_{i=1}^{N} x_i \leq 1$). Table 3 shows that excluding currency hedging leads to a decline in foreign equities and bonds. Overall the equity exposure is clearly reduced.

τ	0.1	0.16	0.2
B CHF	11.9%	19.0%	23.8%
E CHF	4.3	7.0	8.7
C EURO	0.0	0.0	0.0
B EURO	9.4	15.0	18.7
E EURO	1.7	2.7	3.4
C USD	0.0	0.0	0.0
B USD	0.0	0.0	0.0
E USD	9.5	15.1	18.9
C GBP	0.0	0.0	0.0
B GBP	0.0	0.0	0.0
E GBP	1.8	2.8	3.5
C JPY	0.0	0.0	0.0
B JPY	0.0	0.0	0.0
E JPY	3.1	4.9	6.2
E EMMA	0.0	0.0	0.0
C CHF*	58.4	33.3	16.8
Total	100.0	100.0	100.0
Equities	20.4	32.6	40.7
Bonds	21.3	34.0	42.5
Net CH Cash*	58.4	33.3	16.8
Total	100.0	100.0	100.0
Currency Hedging*	0.0	0.0	0.0
μ_x	5.7	6.6	7.1
σ_x	3.7	6.0	7.5

* See table 2

Table 3: No borrowing, no short selling, no currency hedging

Finally, table 4 contains the optimal portfolios under the following constraints:

(1) No short selling of equities and bonds:

$$\begin{aligned} &x_{\mathrm{E\,CHF}}, x_{\mathrm{E\,EURO}}, x_{\mathrm{E\,USD}}, x_{\mathrm{E\,GBP}}, x_{\mathrm{E\,JPY}}, x_{\mathrm{E\,Emma}}, &\geq 0 \\ &x_{\mathrm{B\,CHF}}, x_{\mathrm{B\,EURO}}, x_{\mathrm{B\,USD}}, x_{\mathrm{B\,GBP}}, x_{\mathrm{B\,JPY}} &\geq 0 \end{aligned} \tag{21}$$

(2) Currency hedging is allowed, but overhedging is excluded:

$$\begin{aligned} &x_{\mathrm{E\,EURO}} + x_{\mathrm{B\,EURO}} + x_{\mathrm{C\,EURO}} &\geq 0 \\ &x_{\mathrm{E\,USD}} + x_{\mathrm{E\,Emma}} + x_{\mathrm{B\,USD}} + x_{\mathrm{C\,USD}} &\geq 0 \\ &x_{\mathrm{E\,GBP}} + x_{\mathrm{B\,GBP}} + x_{\mathrm{C\,GBP}} &\geq 0 \\ &x_{\mathrm{E\,JPY}} + x_{\mathrm{B\,JPY}} + x_{\mathrm{C\,JPY}} &\geq 0 \end{aligned} \tag{22}$$

(3) No long position in foreign cash:

$$x_{\mathrm{C\,EURO}}, \ x_{\mathrm{C\,USD}}, \ x_{\mathrm{C\,GBP}}, \ x_{\mathrm{C\,JPY}} \leq 0 \tag{23}$$

(4) No net borrowing in domestic cash:

$$x_{\mathrm{C\,CHF}} + x_{\mathrm{C\,EURO}} + x_{\mathrm{C\,USD}} + x_{\mathrm{C\,GBP}} + x_{\mathrm{C\,JPY}} \geq 0 \tag{24}$$

(5) At least one half of the overall equity investment has to be domestic (home bias):

$$x_{\mathrm{E\,CHF}} - (x_{\mathrm{E\,EURO}} + x_{\mathrm{E\,USD}} + x_{\mathrm{E\,GBP}} + x_{\mathrm{E\,JPY}} + x_{\mathrm{E\,Emma}}) \geq 0 \tag{25}$$

Comparing the results of tables 2 and 4 one observes that the home bias leads to a strong move from foreign (in particular European) equities towards domestic equities. The overall equity exposure sharply declines because the diversification potential is reduced.

τ	0.1	0.16	0.2
B CHF	0.0%	0.0%	0.0%
E CHF	10.3	16.5	20.7
C EURO	–2.0	–3.2	8.3
B EURO	9.0	14.4	6.9
E EURO	0.0	0.0	0.0
C USD	–23.7	–38.0	–41.4
B USD	19.1	30.7	31.7
E USD	6.7	10.7	14.3
C GBP	–1.4	–2.2	–5.4
B GBP	4.7	7.6	11.8
E GBP	0.0	0.0	0.0
C JPY	–4.9	–7.8	–6.1
B JPY	6.0	9.6	8.2
E JPY	1.9	3.0	3.6
E EMMA	1.7	2.8	2.8
C CHF*	72.5	56.0	44.7
Total	100.0	100.0	100.0
Equities	20.6	33.0	41.4
Bonds	38.9	62.3	58.6
Net CH Cash*	40.5	4.7	0.0
Total	100.0	100.0	100.0
Currency Hedging*	32.0	51.3	44.7
μ_x	5.8	6.7	7.2
σ_x	3.8	6.1	7.4

* See table 2

Table 4: Home bias, currency hedging, no borrowing, no short selling

Since a continuous time model is used, one can easily calculate the quantiles for the wealth W_{t,x^*} at time t under a portfolio choice $\boldsymbol{x}^*$. According to formula (9) the α-percentile of W_{t,x^*} is given by

$$W_{t,x^*}(\alpha) = W_0 \cdot \exp\left\{\left(\mu_{x^*} - \frac{\sigma_{x^*}^2}{2}\right)t + \sqrt{t} \cdot \sigma_{x^*} \cdot z_\alpha\right\}, \tag{26}$$

where z_α is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\alpha} e^{-\frac{x^2}{2}}\, dx = \alpha$$

In figure 5 the percentiles of W_{t,x^*} are shown for the optimal portfolio $\boldsymbol{x}^*$ in table 4 corresponding to $\tau = 0.2$ ($\mu_{x^*} = 7.2\,\%$, $\sigma_{x^*} = 7.4\,\%$, $W_0 = 100$).

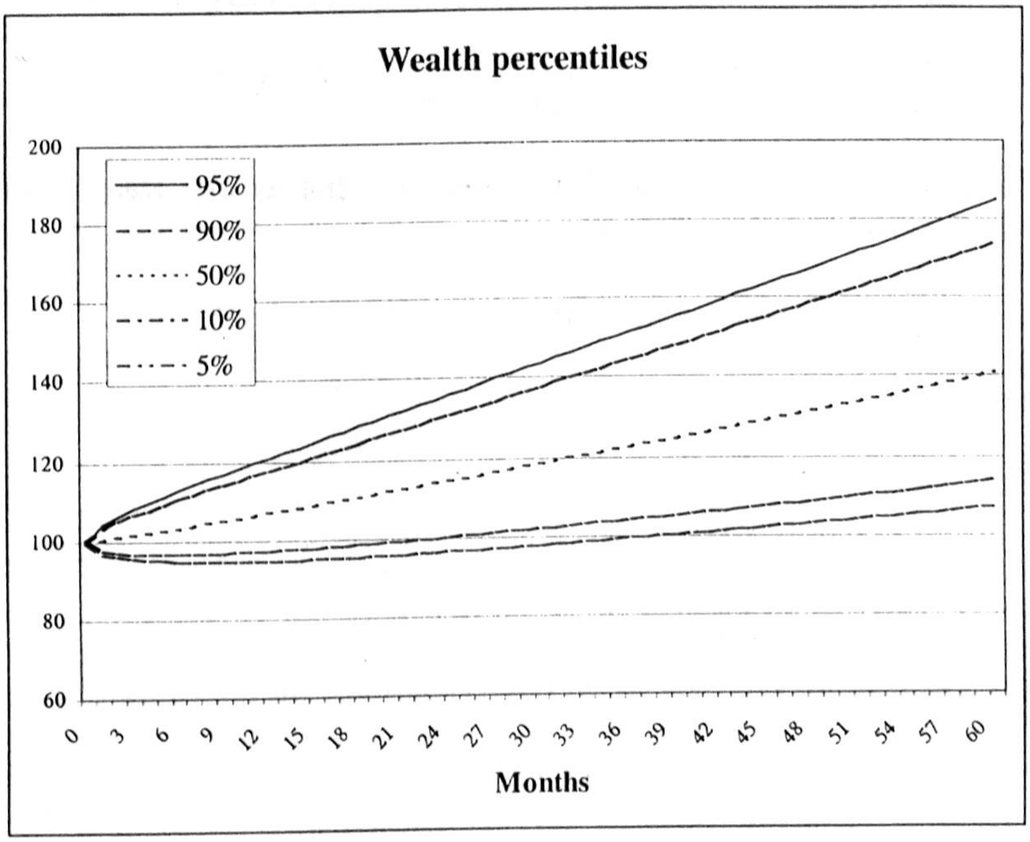

Figure 5: Wealth percentiles for a portfolio $\boldsymbol{x}^*$

3 The Asset Liability Continuous Time Model

3.1 Theoretical Aspects

As in section 2 there is a riskless asset $i = 0$ and N risky investment opportunities. The price processes $S_{i,t}$ are defined as in section 2.
Moreover, there is a process L_t representing a liability. For a pension fund L_t would correspond to the value in t of present and future net obligations. The processes (see also Browne (1999)) are given by

$$\begin{aligned} \frac{dS_0}{S_0} &= r\,dt \\ \frac{dS_i}{S_i} &= \mu_i\,dt + \sigma_i\,dZ_i\,, \quad i = 1, \dots, N \end{aligned} \tag{27}$$

$$\frac{dL}{L} = \mu_L\,dt + \sigma_L\,dZ_L \tag{28}$$

with the Wiener processes $Z_1(t), \dots, Z_N(t)$, $Z_L(t)$ satisfying

$$\begin{aligned} \mathrm{Cov}(\sigma_i\,dZ_i\,, \sigma_j\,dZ_j) &= V_{ij}\,dt \\ \mathrm{Cov}(\sigma_i\,dZ_i\,, \sigma_L\,dZ_L) &= \gamma_i\,dt \end{aligned} \qquad i, j = 1, \dots, N\,. \tag{29}$$

Using the notation of section 2 an investment strategy $\boldsymbol{x}(t)$ leads again to the wealth process W_t (see (8)) given by

$$\frac{dW_t}{W_t} = \mu_{x(t)}\,dt + \sigma_{x(t)}\,dZ_W \tag{30}$$

with

$$\begin{aligned} \mu_{x(t)} &= r + \boldsymbol{\pi}^T \boldsymbol{x}(t) \\ \sigma^2_{x(t)} &= \boldsymbol{x}^T(t) V \boldsymbol{x}(t)\,. \end{aligned}$$

In the following we concentrate on

$$F_t = \frac{W_t}{L_t}\,. \tag{31}$$

Comments:

1. For a pension fund F_t obviously denotes the funding ratio.
2. If L_t is a price index then F_t denotes real wealth (see Adler and Dumas (1983)).

3. Some long term investors may be interested in the performance of their investment strategy $\boldsymbol{x}(t)$ relative to a fixed income portfolio with a price process L_t.

Generally investors may be interested in beating some moving target L_t (pension fund obligations, price level, reference strategy, etc.). In this case it may be appropriate to maximize the expected utility of $F_T = W_T/L_T$. Therefore we look at the asset liability optimization problem (see also Browne (1999))

$$\max_{x(t)\in\mathbb{R}^N,\, t\in[0,T]} E[u(F_T)] \quad \text{subject to} \quad \begin{aligned} A\boldsymbol{x}(t) &\geq \boldsymbol{b} \\ C\boldsymbol{x}(t) &= \boldsymbol{d}\,. \end{aligned} \tag{OAL}$$

As in section 2 the relative risk aversion is assumed to be constant. Hence we deal only with utility functions satisfying (7).
Again it can be shown (Müller (2000)) that (OAL) leads to a constant proportion investment strategy $\boldsymbol{x}$. Therefore F_t is given by a geometric Brownian motion and as shown in the appendix one obtains (see also Müller (1999) or Seiler (2000))

$$\begin{aligned} F_T &= F_0 \cdot \exp\left\{ \left(\mu_{AL,x} - \frac{\sigma^2_{AL,x}}{2} \right) T + \sigma_{AL,x}[Z(T) - Z(0)] \right\} \\ \text{with} \quad & \begin{aligned} \mu_{AL,x} &= \mu_x - \mu_L - \gamma^T \boldsymbol{x} + \sigma_L^2 \\ \sigma^2_{AL,x} &= \sigma_x^2 - 2\gamma^T \boldsymbol{x} + \sigma_L^2\,. \end{aligned} \end{aligned} \tag{A.1}$$

Using these results (see appendix A.2) one can reduce (OAL) to

$$\boxed{\begin{aligned} &\max_{x\in\mathbb{R}^N} \{2\tau\boldsymbol{\pi}^T\boldsymbol{x} - \boldsymbol{x}^T V\boldsymbol{x} + 2(1-\tau)\gamma^T\boldsymbol{x}\}, \quad \text{with } \tau = \frac{1}{c} \\ &\text{subject to} \quad \begin{aligned} A\boldsymbol{x} &\geq \boldsymbol{b} \\ C\boldsymbol{x} &= \boldsymbol{d}\,. \end{aligned} \end{aligned}} \tag{OALS}$$

1. (OALS) is not fully analogous to the one period surplus optimization. In fact the continuous time asset liability model is based on the ratio of final wealth W_T and the liability L_T and not on their difference. The impact of the liability on the portfolio choice decreases for high levels of risk tolerance and disappears for the logarithmic utility function ($\tau = 1$).
2. Without constraints one obtains a well known mutual fund theorem. The optimal portfolios are given by

$$\boldsymbol{x}^* = \tau \underbrace{V^{-1}\boldsymbol{\pi}}_{\substack{\text{growth optimum} \\ \text{portfolio}}} + (1-\tau)V^{-1}\gamma\,, \quad \tau > 0\,. \tag{32}$$

3.2 Practical Implementation

a. Application of the Asset Liability Model

Typically the continuous time asset liability model is used for pension funds and individual investors with a long investment horizon. The liability is represented by a fixed income portfolio. Implicitly one assumes that these investors always plan a fixed time horizon ahead.
Moreover, the model can be used for investors oriented towards several currencies. In this case the liability is given by a multicurrency cash portfolio. Sometimes one applies the asset liability model for investors interested in real rather than in nominal wealth. As a liability one uses the consumer price index. However, in most cases the impact on the portfolio choice is small and it is questionable to represent the consumer price index by a geometric Brownian motion.

b. Estimation of V, γ, π and τ

Variances and covariances (V, γ) can be estimated as in section 2. For the risk premia (π) we shall use the estimation of table 1.
For the estimation of the risk tolerance τ one can proceed as in section 2. In the absence of constraints the optimality conditions of (OALS) are given by

$$\tau\pi - V\boldsymbol{x}^* + (1 - \tau)\gamma = \mathbf{0} \tag{33}$$

which can be transformed into

$$\tau(\pi - \gamma)^T \boldsymbol{x}^* = \boldsymbol{x}^{*T} V \boldsymbol{x}^* - \gamma^T \boldsymbol{x}^* \tag{34}$$

or

$$\tau = \frac{\boldsymbol{x}^{*T} V \boldsymbol{x}^* - \gamma^T \boldsymbol{x}^*}{(\pi - \gamma)^T \boldsymbol{x}^*} . \tag{35}$$

Again, given some idea about the compostion of $\boldsymbol{x}^*$, one can get a rough estimation of τ.

c. Calculation of Optimal Portfolios

In order to illustrate the asset liability model we look at an investor with a long investment horizon who is oriented towards Swiss francs (65 %), EURO (25 %) and US dollar (10 %). Hence his reference strategy L_t is given by a portfolio consisting of 65 % Swiss bonds, 25 % EURO bonds and 10 % US bonds. Given the covariance matrix V estimated in section 2 one can immediately calculate γ. First, in analogy to table 2 we solve OALS in the absence of constraints (table 6). Of course in the extreme cases $\tau = 1$, $\tau = 0$, the optimum growth and the liability tracking portfolios, respectively, result. For the intermediate values of τ combinations of the optimum growth and the liability tracking portfolio are obtained in accordance to (32).

τ	0	0.1	0.16	0.2	0.3	1
B CHF	65.0%	58.7%	54.8%	52.3%	46.0%	1.6%
E CHF	0.0	0.7	1.2	1.4	2.2	7.2
C EURO	0.0	–14.4	–23.0	–28.7	–43.1	–143.7
B EURO	25.0	34.9	40.9	44.9	54.8	124.5
E EURO	0.0	4.7	7.5	9.3	14.0	46.7
C USD	0.0	–24.2	–38.8	–48.5	–72.7	–242.4
B USD	10.0	24.0	32.5	38.1	52.1	150.5
E USD	0.0	12.8	20.5	25.6	38.4	128.0
C GBP	0.0	–3.7	–5.9	–7.3	–11.0	–36.6
B GBP	0.0	1.9	3.1	3.9	5.8	19.3
E GBP	0.0	2.4	3.9	4.9	7.3	24.3
C JPY	0.0	–9.4	–15.1	–18.8	–28.3	–94.2
B JPY	0.0	7.9	12.6	15.8	23.6	78.8
E JPY	0.0	3.3	5.3	6.7	10.0	33.4
E EMMA	0.0	1.0	1.6	2.0	3.1	10.2
C CHF*	0.0	–0.7	–1.2	–1.5	–2.2	–7.4
Total	100.0	100.0	100.0	100.0	100.0	100.0
Equities	0.0	25.0	40.0	49.9	74.9	249.7
Bonds	100.0	127.5	144.0	154.9	182.4	374.6
Net CH Cash*	0.0	–52.4	–84.0	–104.9	–157.3	–524.3
Total	100.0	100.0	100.0	100.0	100.0	100.0
Currency Hedging*	0.0	51.7	82.8	103.4	155.1	517.0
μ_x	5.0	6.5	7.4	8.0	9.6	20.2
σ_x	3.1	6.0	8.1	9.5	13.3	39.8

* See table 2

Table 6: Liabilities, unconstrained

In analogy to table 3 we solve OALS for the case where currency hedging, short selling and borrowing is excluded (i. e. $x_i \geq 0$, $i = 1, \ldots, N$, $\sum_{i=1}^{N} x_i \leq 1$).

Optimization leads to portfolios which differ substantially from the asset only results in table 3. Comparing tables 3 and 7 we observe a strong increase in bond holdings, which is caused by the liabilities. The small increase in overall equity holdings may be explained by the fact that the liability tracking portfolio consists of bonds and there are positive correlations (covariances) between equities and bonds.

τ	0.1	0.16	0.2
B CHF	44.3%	29.6%	19.9%
E CHF	4.0	7.1	9.1
C EURO	0.0	0.0	0.0
B EURO	31.8	34.1	35.6
E EURO	1.7	2.7	3.4
C USD	0.0	0.0	0.0
B USD	0.1	0.0	0.0
E USD	13.4	18.8	22.2
C GBP	0.0	0.0	0.0
B GBP	0.1	0.0	0.0
E GBP	2.1	3.6	4.5
C JPY	0.0	0.0	0.0
B JPY	0.0	0.0	0.0
E JPY	2.5	4.2	5.3
E EMMA	0.0	0.0	0.0
C CHF*	0.0	0.0	0.0
Total	100.0	100.0	100.0
Equities	23.7	36.2	44.5
Bonds	76.3	63.8	55.5
Net CH Cash*	0.0	0.0	0.0
Total	100.0	100.0	100.0
Currency Hedging*	0.0	0.0	0.0
μ_x	6.3	7.0	7.5
σ_x	5.4	7.3	8.6

* See table 2

Table 7: Liabilities, no short selling, no currency hedging

Finally, table 8 contains the asset liability results under the constrains (21)–(25) imposed in section 2.

τ	0.1	0.16	0.2
B CHF	23.2%	0.0%	0.0%
E CHF	10.8	17.2	21.4
C EURO	–0.5	1.0	15.2
B EURO	30.6	32.4	21.4
E EURO	0.0	0.0	0.0
C USD	–10.5	–16.6	–19.6
B USD	14.6	17.1	17.3
E USD	8.3	13.3	17.0
C GBP	–6.3	–10.2	–13.7
B GBP	7.7	12.6	17.1
E GBP	0.0	0.0	0.0
C JPY	–1.4	–1.9	0.5
B JPY	2.4	3.5	1.4
E JPY	1.8	2.8	3.4
E EMMA	0.7	1.1	1.0
C CHF*	18.6	27.7	17.6
Total	100.0	100.0	100.0
Equities	21.5	34.5	42.8
Bonds	78.5	65.5	57.2
Net CH Cash*	0.0	0.0	0.0
Total	100.0	100.0	100.0
Currency Hedging*	18.6	27.7	17.6
μ_x	6.3	7.0	7.5
σ_x	5.3	7.2	8.4

* See table 2

Table 8: Liabilities, home bias, currency hedging, no borrowing, no short selling

Due to the home bias there are important positions in Swiss equities. Despite the strong weight of Swiss bonds in the liability tracking portfolio (65 %), Swiss bonds do not occur for $\tau = 0.160$ and $\tau = 0.2$. This contrast to table 7 may be explained by substitution effects between Swiss bonds and Swiss equities.

4 Conclusions

In the asset only case there are strong similarities between one period Markowitz optimization and the continuous time model under consideration. Since in the continuous time model the asset price processes are given by geometric Brownian motions it is fully compatible with Black/Scholes option pricing. Moreover, under the optimal investment policy, final wealth is given as well by a geometric Brownian motion and is therefore lognormally distributed in each point of time. Hence its quantiles can be easily calculated. This contrasts with the rather messy distribution of final wealth resulting from a sequential application of the one period Markowitz method.

In the asset liability case, for the following reason there is no full analogy between the one period and the continuous time model: The one period model deals with the surplus, i. e. the difference between final wealth and some stochastic reference level (such as value of liabilities, initial wealth adjusted to inflation, etc.). The continuous time model, however, puts its emphasis on the corresponding ratio (e. g. funding ratio of a pension fund, final wealth in real terms, etc.). In practice the continuous time model can be applied to investors with a long investment horizon or with a multicurrency objective. Their final wealth is measured in terms of a fixed income or a multicurrency portfolio.

Appendix: Formulas for the asset liability model

Appendix A.1

Applying Itô's lemma to $F_t = W_t/L_t$ leads to

$$\frac{dF}{F} = \frac{dW}{W} - \frac{dL}{L} - \frac{dW}{W} \cdot \frac{dL}{L} + \left(\frac{dL}{L}\right)^2.$$

Using (27)–(30) one obtains

$$\frac{dF}{F} = (\mu_x - \mu_L - \boldsymbol{\gamma}^T \boldsymbol{x} + \sigma_L^2)dt + \sum_{i=1}^{N} x_i \sigma_i \, dZ_i - \sigma_L \, dZ_L$$

or

$$\frac{dF}{F} = \mu_{AL,x} \, dt + \sigma_{AL,x} \, dZ$$

with

$$\mu_{AL,x} = \mu_x - \mu_L - \boldsymbol{\gamma}^T \boldsymbol{x} + \sigma_L^2$$
$$\sigma_{AL,x}^2 = \boldsymbol{x}^T V \boldsymbol{x} - 2\boldsymbol{\gamma}^T \boldsymbol{x} + \sigma_L^2$$
$$dZ \sim N(0, \sqrt{dt})\,.$$

From this A.1 follows immediately (see e. g. Hull (1999)).

Appendix A.2

For $c \neq 1$ one obtains from A.1

$$u(F_T) = (1-c)F_0^{1-c} \exp\left\{(1-c)\left(\mu_{AL,x} - \frac{\sigma_{AL,x}^2}{2}\right)T + (1-c)\sigma_{AL,x}[Z(T) - Z(0)]\right\}.$$

In analogy to (10)–(12) this leads to

$$E[u(F_T)] = (1-c)F_0^{1-c} \exp\left\{(1-c)\left[\mu_{AL,x} - \frac{c}{2}\sigma_{AL,x}^2\right]T\right\}.$$

Maximization of $E\{u(F_T)\}$ is equivalent to the maximization of

$$\mu_{AL,x} - \frac{c}{2}\sigma_{AL,x}^2$$

or

$$\frac{2}{c}(\mu_x - \boldsymbol{\gamma}^T \boldsymbol{x}) - \boldsymbol{x}^T V \boldsymbol{x} + 2\boldsymbol{\gamma}^T \boldsymbol{x}$$

or (due to (30))

$$2\tau\boldsymbol{\pi}^T \boldsymbol{x} - \boldsymbol{x}^T V \boldsymbol{x} + 2(1-\tau)\boldsymbol{\gamma}^T \boldsymbol{x}\,, \quad \text{with } \tau = \frac{1}{c}\,.$$

The case $c = 1$ can be dealt with similarly.

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Abstract

The goal of this paper is to show a practical application of creating benchmark portfolios using continuous time assumptions. If stocks follow a geometric Brownian motion and investors have a constant relative risk aversion profile, one can apply Merton's findings that the investment decision can be solved using static optimization. This result also holds given constraints on the portfolio weights. The technique of "reverse optimization" to calculate the risk premiums and estimation of the covariance matrix using historical data is explained. A few examples are given and the continuous time assumptions enable one to easily display the risk and return characteristics of the investment strategy over time with the help of distribution quantiles.

An asset-liability model in continuous time is also presented. For example with pension funds it is typical to consider the future funding ratio (ratio of assets over liabilities) instead of end of period wealth. Similarly, a private investor maximizing real wealth would optimize the quotient of nominal end of period wealth over a price index. According to these examples the quotient of end of period wealth over a stochastic benchmark index is optimized. This problem simplifies to a static optimization as well.

Zusammenfassung

Ziel der vorliegenden Arbeit ist die praktische Anwendung der Portfoliotheorie in kontinuierlicher Zeit zur Bestimmung von Benchmark Portfolios. Falls die Kursentwicklungen durch geometrische Brownsche Bewegungen gegeben sind und konstante relative Risikoaversion vorliegt, so gilt auch unter Anlagerestriktionen das bekannte Resultat von Merton, demgemäss sich das Anlageproblem auf eine statische Optimierung zurückführen lässt. In der Arbeit wird auf die Schätzung der Kovarianzmatrix mittels historischer Daten und die "Reverse Optimization"-Technik zur Ermittlung der Risikoprämien eingegangen. Anschliessend werden einige Beispiele gerechnet. Der Ansatz in kontinuierlicher Zeit erlaubt es, anhand von Quantilen das Rendite/Risiko-Verhältnis einer Anlagestrategie im zeitlichen Ablauf einfach darzustellen.

Zudem wird ein "Asset Liability"-Ansatz in kontinuierlicher Zeit vorgestellt. Bei einer Pensionskasse ist es naheliegend, anstelle des Endvermögens, den zukünftigen Deckungsgrad (Quotient aus Vermögen und Wert der Verbindlichkeiten) zu optimieren. Analog ist bei einem privaten Investor mit realer Zielsetzung das reale Endvermögen, d. h. der Quotient aus nominellem Endvermögen und Preisindex, zu optimieren. Ausgehend von diesen Beispielen wird beim vorgestellten "Asset Liability"-Ansatz der Quotient vom Endvermögen und einer stochastischen Referenzgrösse optimiert. Auch dieses Problem lässt sich auf eine statische Optimierung zurückführen.

Résumé

L'objectif de ce papier est de présenter une application pratique de la théorie du portefeuille en temps continu pour construire des portefeuilles de référence (benchmarkportfolios). Lorsque la valeur des actifs suit un mouvement brownien géométrique et que les investisseurs ont une aversion au risque constante le résultat de Merton, selon lequel le problème de l'investisseur est réduit à une optimisation statique, peut être appliqué même si les proportions des divers portefeuilles sont soumises à des restrictions. Dans ce travail l'estimation de la matrice de covariance à l'aide de

données historiques et la détermination de la prime de risque avec la technique de la "reverse optimization" sont expliquées. On donne quelques exemples.
L'hypothèse du temps continu permet de représenter facilement les rapports rendement/risque de la stratégie d'investissement dans le temps au moyen de quantiles.
On présente également un modèle actif/passif en temps continu. Pour une caisse de pension il est logique d'optimiser le degré de couverture (quotient de la valeur des actifs par la valeur des garanties) au lieu de la fortune finale. De manière analogue, un investisseur privé, dans le but de maximaliser sa fortune finale, optimisera le quotient de sa fortune finale nominale par un indice des prix. En accord avec ces exemples on optimise le quotient de la fortune finale par un indice aléatoire. Ce problème se réduit également à une optimisation statique.