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The severity of ruin in Markov-modulated risk models

1 Introduction

The theory of ruin has been the object of inspiration of several authors. A huge related literature is central to the most existing works and papers [c.f. Bowers et al. (1986), Gerber (1979), Grandell (1991)]. In recent years a great variety of results on evaluation of ruin probabilities has been studied in various kinds of risk models, see for example Grandell (1991) and references there in. In this paper we are interested in the severity of ruin in Markov-modulated model. Models of this type have been investigated, e.g., by Asmussen (1989), Reinhard (1984) who were concerned in finding the probability of ruin in finite or infinite time as function of the initial reserve. Bäuerle (1996) has been interested in the expected ruin time.

Considerable attention has been given to the probability of ruin, because it is widely considered as a powerful tool for the control of risk and the determination of stability criterions. The classical results concerning ruin probabilities were obtained by Arfwedson (1950), Beekman (1966), Cramér (1955), Prabhu (1961) and generalized by Cai and Wu (1997), Dufresne and Gerber (1991), Janssen (1981, 1982), Janssen and Reinhard (1985), Reinhard (1984), Thorin (1975), Willmot (1994) and Wu (1999), ...

Unfortunately, this probability is not completely satisfactory in some cases, for instance it has been pointed out by Gerber, Goovaerts and Kaas (1987) that it is a "very crude stability criterion".

In order to fill these deficiencies, we will introduce the severity of ruin. The practical interest of this concept is, in particular, the possibility of bringing an additional element of information on the ruin.

The severity of ruin was first studied by Gerber et al. (1987) and Dufresne and Gerber (1988), in the classical risk model, where the risk process is a compound Poisson process and the premium rate is constant. Then, Dufresne (1989) extended this concept to model where a diffusion process is added to the compound Poisson process, the generalization can be interpreted as allowing for some uncertainty both in the premium income and in the claim amounts. We also emphasize the considerable works dealing with the severity of the ruin in a discrete risk model. References we may cite on this topic are Dickson and Waters (1992), Dickson et al. (1995), Gerber (1988) and Reinhard (1997). More recently Reinhard and Snoussi (1998) have extended the results in a discrete semi-Markov risk model.

The aim of this paper is to study this distribution in Markov-modulated model where the intensity and the premium can fluctuate according to a Markovian environment.
In Section 2 basic definitions and results in the Markov-modulated model are provided. Then, in Section 3 a differential system for the severity of ruin is established. Finally, in Section 4, we will calculate explicitly the severity of ruin in the particular case where one has two possible states for the environment process and certain types of claim size distribution.

2 Preliminaries

In this model we suppose that the frequencies of the claims and their amounts are influenced by an external environment $\{I(t),\ t \geq 0\}$ evolving randomly in a space of m states $1, \ldots, m$ $(m \in \mathbb{N}_0)$. As pointed out by Asmussen (1989), in health insurance, sojourns of $\{I(t),\ t \geq 0\}$ could be a certain type of epidemics or, in automobile insurance, could be the weather type (for example, icy, foggy, ...). We suppose that $\{I(t),\ t \geq 0\}$ is a homogeneous, irreducible and recurrent Markov process with finite state space $J = \{1, \ldots, m\}$. We denote by $\pi = (\pi_1, \ldots, \pi_m)$ its unique stationary probability distribution and by $\Lambda = (\alpha_{ij})$ where $\alpha_{ii} = -\alpha_i$ the intensity matrix. The transition probability matrix of the embedded Markov chain is given by

$$p_{ij} = \begin{cases} 0 & \text{if } i = j \\ \dfrac{\alpha_{ij}}{\alpha_i} & \text{if } i \neq j \end{cases}$$

Moreover, we suppose that at time t claims occur according to a Poisson process with intensity $\lambda_i \in \mathbb{R}^+$ if $I(t) = i$ and the corresponding claim sizes have distributions $F_i(x)$ with finite mean μ_i $(i \in J)$. Assume further, that premiums are received continuously at a constant rate $c_i > 0$ during any time interval when the environment process remains in state i. Denote now by A_n the time between the $(n-1)-st$ and $n-th$ claim arrival and by B_n the amount of the $n-th$ claim.
We will suppose that the sequences of random variables (A_n) and (B_n) are conditionally independent given $\{I(t),\ t \in \mathbb{R}^+\}$.
Let $N(t) = \sup\{n \in \mathbb{N} \mid \sum_{k=1}^{n} A_k \leq t\}$ be the number of claims that have occurred before time t. The insurer's surplus $\{Z(t), t \geq 0\}$ is given by

$$Z(t) = u + C(t) + \sum_{n=1}^{N(t)} B_n \,,$$

where $C(t)$ denotes the total premium received during $(0, t]$ and $u \in \mathbb{R}^+$ the insurer's initial surplus. Reinhard (1984) showed that

$$C(t) = \sum_{k=1}^{N_e(t)} c_{I_{k-1}} \cdot (T_k - T_{k-1}) + c_{I_{N_e(t)}} (t - T_{N_e(t)})$$

where I_k and T_k denote respectively the state of the environment after its $k - th$ transition and the time at which occurs the $k - th$ transition of the environment process, and $N_e(t) = \sup\{k : T_k \leq t\}$.
We define the ultimate ruin probabilities by

$$\psi_i(u) = \mathbb{P}(T < \infty \mid Z(0) = u, \; I(0) = i) \tag{1}$$

where T denotes the time of ruin,

$$T = \inf\{t > 0 \mid Z(t) < 0\}$$

($T = \infty$ if ruin does not occur). The corresponding ultimate survival probabilities

$$R_i(u) = 1 - \psi_i(u) \tag{2}$$

It is well known from the theory of random walks (see Reinhard (1984)), that

- $\psi_i(u) > 0 \; \forall u \geq 0, \; i \in J$ and $\psi_i(\infty) = 0$ if $d = \sum_{j=1}^{m} \pi_j \left(\frac{c_j}{\lambda_j} - \mu_j \right)$ is strictly positive.
- $\psi_i(u) = 1 \; \forall u \geq 0, \; i \in J$ if $d \leq 0$.

We suppose therefore that $d > 0$.

3 Properties of the severity of ruin

We define the severity of the ruin in Markov-modulated risk models, in the following way

$$\psi_i(u, y) = \mathbb{P}(T < \infty \text{ and } Z_T < -y \mid Z_0 = u, I(0) = i) \qquad (u, y \in \mathbb{R}^+) \tag{3}$$

This represents the probability that ruin occur and that at ruin time the surplus takes a value less then $-y$ starting from an initial surplus u, and an initial environment $i \in J$.

Note that by letting $y = 0$ we obtain (1), i.e. $\psi_i(u, 0) = \psi_i(u)$. The method used hereafter for computing the probabilities $\psi_i(u, y)$ may also be used for computing the probabilities

$$\begin{aligned} G_i(u, y) &= \mathbb{P}(T < \infty \quad \text{and} \quad Z_T \geq -y \mid Z_0 = u,\ I(0) = i) \\ &= \psi_i(u) - \psi_i(u, y) \qquad\qquad (u, y \in \mathbb{R}^+) \end{aligned} \tag{4}$$

The probabilities (4) generalize those introduced by Gerber et al. (1987) in the classical risk model.
In the following theorem, we derive a differential system for $\psi_i(u, y)$.

Theorem 1 *For all $i \in J$, the probabilities of the severity of ruin satisfy the following system of differential equations*

$$\begin{aligned} c_i \frac{\partial}{\partial u} \psi_i(u, y) &= (\alpha_i + \lambda_i)\psi_i(u, y) \\ &\quad - \lambda_i \left[\int_0^u \psi_i(u - x, y)\, dF_i(x) + 1 - F_i(u + y) \right] \\ &\quad - \alpha_i \sum_{k=1}^m p_{ik} \psi_k(u, y) \end{aligned} \tag{5}$$

Proof Conditioning on what happens in a small time interval $[0, h]$ with $h > 0$, we get

$$\begin{aligned} \psi_i(u, y) &= (1 - \alpha_i h - \lambda_i h)\psi_i(u + c_i h, y) \\ &\quad + \lambda_i h \left[\int_0^{u + c_i h} \psi_i(u + c_i h - x, y)\, dF_i(x) + 1 - F_i(u + c_i h + y) \right] \\ &\quad + \alpha_i h \sum_{k=1}^m p_{ik} \psi_k(u + c_i h, y) + o(h) \end{aligned} \tag{6}$$

(where $\frac{o(h)}{h} \xrightarrow{h \to 0} 0$). The right hand side can be interpreted as follows:
the first term corresponds to the case where no claim and no change of environment occur in the interval $[0, h]$, the second term to the case where a claim occurs in $[0, h]$ (it can either cause the ruin or not), the third term to the case where the environment changes in $[0, h]$. Finally, the possibility that two or more events occur in $[0, h]$ is a $o(h)$.

Then we get

$$\frac{\psi_i(u+c_ih,y)-\psi_i(u,y)}{h}$$
$$= (\alpha_i+\lambda_i)\psi_i(u+c_ih,y)$$
$$-\lambda_i\Big[\int_0^{u+c_ih}\psi_i(u+c_ih-x,y)dF_i(x)+1-F_i(u+c_ih+y)\Big]$$
$$-\alpha_i\sum_{k=1}^{m}p_{ik}\psi_k(u+c_ih,y)+\frac{o(h)}{h}$$

The desired result follows by letting h tend to 0. ■

We can also show that (5) has a unique solution such that $\psi_i(\infty,y)=0$ $(i\in J,\ y\in\mathbb{R}^+)$

Proposition 1

$$\psi_i(t,y)=\psi_i(0,y)+\frac{\lambda_i}{c_i}\int_{u=0}^{t}\psi_i(t-u,y)(1-F_i(u))\,du$$
$$-\frac{\lambda_i}{c_i}\int_0^t(1-F_i(u+y))\,du$$
$$+\frac{\alpha_i}{c_i}\int_0^t\Big[\psi_i(u,y)-\sum_{k=1}^{m}p_{ik}\psi_k(u,y)\Big]\,du \tag{7}$$

Proof by integrating (5) over $(0,t)$, it yields:

$$c_i\,\psi_i(t,y)=c_i\,\psi_i(0,y)+(\lambda_i+\alpha_i)\int_0^t\psi_i(u,y)\,du$$
$$-\lambda_i\int_0^t(1-F_i(u+y))\,du$$
$$-\lambda_i\int_{u=0}^{t}\int_{x=0}^{u}\psi_i(u-x,y)\,dF_i(x)\,du$$
$$-\alpha_i\sum_{k=1}^{m}p_{ik}\int_0^t\psi_i(u,y)\,du$$

Thus, we have

$$\begin{aligned}
\psi_i(t,y) &= \psi_i(0,y) + \frac{\lambda_i}{c_i}\int_0^t \psi_i(u,y)\,du \\
&\quad - \frac{\lambda_i}{c_i}\int_{x=0}^{t}\int_{z=0}^{t-x} \psi_i(z,y)\,dz\,dF_i(x) \\
&\quad - \frac{\lambda_i}{c_i}\int_0^t (1-F_i(u+y))\,du \\
&\quad + \frac{\alpha_i}{c_i}\int_0^t \left[\psi_i(u,y) - \sum_{k=1}^{m} p_{ik}\psi_k(u,y)\right] du \\
&= \psi_i(0,y) + \frac{\lambda_i}{c_i}\int_0^t \psi_i(t-u,y)\,du \\
&\quad - \frac{\lambda_i}{c_i}\int_0^t (1-F_i(u+y))du \\
&\quad - \frac{\lambda_i}{c_i}\int_{u=0}^{t} \psi_i(t-u,y)F_i(u)\,du \\
&\quad + \frac{\alpha_i}{c_i}\int_0^t \left[\psi_i(u,y) - \sum_{k=1}^{m} p_{ik}\psi_k(u,y)\right] du \qquad \blacksquare
\end{aligned}$$

Note that for $m = 1$, (7) is the well known renewal equation for the severity of ruin in the classical risk model and it is essentially the formula (5.19) in Dufresne (1989), i.e.,

$$\begin{aligned}
\psi(u,y) &= \psi(0,y) + \frac{\lambda}{c}\int_{x=0}^{u} \psi(u-x,y)(1-F(x))\,dx \\
&\quad - \frac{\lambda}{c}\int_0^u (1-F(x+y))\,dx \qquad (8)
\end{aligned}$$

Unfortunately for $m > 1$, (7) is not more a renewal type equation.

Proposition 2

$$\psi_i(0,y) = \frac{\lambda_i}{c_i}\int_y^\infty (1 - F_i(u))\, du$$

$$- \frac{\alpha_i}{c_i}\int_0^\infty \left[\psi_i(u,y) - \sum_{k=1}^m p_{ik}\psi_k(u,y)\right] du \tag{9}$$

Proof the result follows immediately by letting t in (7) tend to infinity. ■

Note that in the case $m = 1$, we obtain the following result derived by Dufresne (1989),

$$\psi(0,y) = \frac{\lambda}{c}\int_y^\infty (1 - F(u))\, du$$

which is equivalent to the result obtained by Bowers et al. (1986) (Theorem 12.2),

$$G(0,y) = \frac{\lambda}{c}\int_0^y (1 - F(u))\, du$$

see also Gerber et al. (1987).

4 The two state model

In this section, we obtain analytic expressions for the severity of ruin. The method used below consist to eliminate the integral term on the right-hand side of (5) by differentiation. This result leads us to change an integrodifferential equation to a differential equation which is easier to solve. This technique works only for certain types of claim size distribution.
We consider the case where $I(t)$ is a two state Markov process (only two environment states). We show that an explicit formula for $\psi_i(u,y)$ can be given under the additional assumption that the claim amounts distribution are exponential with finite mean μ_i.

$$F_i(x) = 1 - e^{-\frac{1}{\mu_i}x} \qquad (x \geq 0)\,.$$

The same argument can be used for a mixture of exponential distributions. We suppose that the premium income is constant and independent from the environment process (i.e. $c_i = c \quad (i = 1, 2)$). Our approach is closely related to that of Reinhard (1984). In this case (5) is reduced to

$$c\frac{\partial}{\partial u}\psi_i(u,y) = (\lambda_i + \alpha_i)\psi_i(u,y) - \frac{\lambda_i}{\mu_i}e^{-\frac{1}{\mu_i}u}\int_{z=0}^{u}\psi_i(z,y)e^{\frac{1}{\mu_i}z}\,dz$$
$$- \lambda_i e^{-\frac{1}{\mu_i}(u+y)} - \alpha_i\psi_{\vartheta(i)}(u,y) \tag{10}$$

(where $\vartheta(1) = 2, \vartheta(2) = 1$). Differentiation of (10) leads to

$$c\frac{\partial^2}{\partial u^2}\psi_i(u,y) = (\lambda_i + \alpha_i)\frac{\partial}{\partial u}\psi_i(u,y) + \frac{1}{\mu_i}\left[\frac{\lambda_i}{\mu_i}\int_{z=0}^{u}\psi_i(z,y)e^{-\frac{1}{\mu_i}(u-z)}\,dz\right]$$
$$- \frac{\lambda_i}{\mu_i}\psi_i(u,y) + \frac{\lambda_i}{\mu_i}e^{-\frac{1}{\mu_i}(u+y)} - \alpha_i\frac{\partial}{\partial u}\psi_{\vartheta(i)}(u,y)$$

Finally, by replacing the term between the brackets, we obtain

$$c\frac{\partial^2}{\partial u^2}\psi_i(u,y) = \left(\lambda_i + \alpha_i - \frac{c}{\mu_i}\right)\frac{\partial}{\partial u}\psi_i(u,y) + \frac{\alpha_i}{\mu_i}\psi_i(u,y)$$
$$- \alpha_i\frac{\partial}{\partial u}\psi_{\vartheta(i)}(u,y) - \frac{\alpha_i}{\mu_i}\psi_{\vartheta(i)}(u,y)$$

with boundary conditions

$$\begin{cases} \psi_i(\infty,y) = 0 \\ c\dfrac{\partial}{\partial u}\psi_i(u,y)\Big|_{u=0} = (\lambda_i + \alpha_i)\psi_i(0,y) \\ \qquad\qquad\qquad -\lambda_i(1 - F_i(y)) - \alpha_i\psi_{\vartheta(i)}(0,y) \end{cases} \tag{11}$$

These differential equations may be written in matrix form

$$\mathcal{Y}' = \mathcal{A}\mathcal{Y}$$

where

$$\mathcal{A} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\alpha_1}{c\mu_1} & -\dfrac{\alpha_1}{c\mu_1} & \dfrac{\alpha_1}{c} - \varrho_1 & -\dfrac{\alpha_1}{c} \\ -\dfrac{\alpha_2}{c\mu_2} & \dfrac{\alpha_2}{c\mu_2} & -\dfrac{\alpha_2}{c} & \dfrac{\alpha_2}{c} - \varrho_2 \end{vmatrix}$$

$$\mathcal{Y} = \left[\psi_1(u,y), \psi_2(u,y), \frac{\partial}{\partial u}\psi_1(u,y), \frac{\partial}{\partial u}\psi_2(u,y)\right]'$$

and

$$\varrho_i = \frac{1}{\mu_i} - \frac{\lambda_i}{c}, \qquad (i = \{1, 2\})$$

Without loss of generality we suppose that $\varrho_1 \geq \varrho_2$. In this case

$$d = \frac{\frac{\alpha_2}{c\mu_2}\varrho_1 + \frac{\alpha_1}{c\mu_1}\varrho_2}{\lambda_2\alpha_1 + \lambda_1\alpha_2}$$

and the condition $d > 0$, is here equivalent to the following :

$$\frac{\alpha_2}{c\mu_2}\varrho_1 + \frac{\alpha_1}{c\mu_1}\varrho_2 > 0$$

As $\varrho_1 \geq \varrho_2$, then ϱ_1 is clearly strictly positive. On the other hand, it is clear that $k_0 = 0$ is an eigenvalue of $\mathcal{A}$ (i.e. $\det(\mathcal{A}) = 0$). The other three eigenvalues k_1, k_2 and k_3 are roots of the following characteristic equation

$$\mathcal{P}(k) = \frac{1}{k}\det(\mathcal{A} - k\mathcal{I}) = 0$$

where

$$\begin{aligned}\mathcal{P}(k) = k^3 &+ \left(\varrho_1 + \varrho_2 - \frac{\alpha_1}{c} - \frac{\alpha_2}{c}\right)k^2 \\ &+ \left((\varrho_1 - \frac{\alpha_1}{c})(\varrho_2 - \frac{\alpha_2}{c}) - \frac{\alpha_1}{c\mu_1} - \frac{\alpha_2}{c\mu_2} - \frac{\alpha_1\alpha_2}{c^2}\right)k \\ &- \left(\varrho_1\frac{\alpha_2}{c\mu_2} + \varrho_2\frac{\alpha_1}{c\mu_1}\right)\end{aligned} \tag{12}$$

From $d > 0$, we have $k_1k_2k_3 > 0$. Since

$$\begin{aligned}&\mathcal{P}(0) < 0 \\ &\mathcal{P}(-\varrho_1) = \frac{\alpha_1\lambda_1}{c^2}(\varrho_1 - \varrho_2) \geq 0 \\ &\mathcal{P}(-\varrho_2) = \frac{\alpha_2\lambda_2}{c^2}(\varrho_2 - \varrho_1) \leq 0 \qquad \text{and} \\ &\lim_{x\to\pm\infty} \mathcal{P}(x) = \pm\infty\end{aligned}$$

it follows from Reinhard (1984, p.41) that

$$\begin{cases} k_1 < -\varrho_1 < k_2 \; < \min\{0, -\varrho_2\}, \; k_3 > 0 & \text{if } \varrho_1 > \varrho_2 \\ k_1 < k_2 = -\varrho < 0 < k_3 & \text{if } \varrho_1 = \varrho_2 = \varrho \end{cases}$$

Therefore the general solution is of the following type

$$\begin{cases} \psi_1(u,y) = A_0 + A_1 e^{k_1 u} + A_2 e^{k_2 u} + A_3 e^{k_3 u} \\ \psi_2(u,y) = A_0 - D(k_1) A_1 e^{k_1 u} - D(k_2) A_2 e^{k_2 u} - D(k_3) A_3 e^{k_3 u} \end{cases}$$

Where

$$D(k_i) = \frac{c k_i^2 \mu_1 - k_i(-c + \mu_1 \alpha_1 + \mu_1 \lambda_1) - \alpha_1}{\alpha_1 + k_i \mu_1 \alpha_1} \tag{13}$$

From (11), we have clearly that $A_0 = A_3 = 0$ and A_1, A_2 are the solutions of the following linear equation

$$\begin{cases} \Big[ck_1 - \lambda_1 - \alpha_1 - \alpha_1 D(k_1)\Big] A_1 \\ \qquad + \Big[ck_2 - \lambda_1 - \alpha_1 - \alpha_1 D(k_2)\Big] A_2 = -\lambda_1 e^{-\frac{1}{\mu_1} y} \\ \Big[(-ck_1 + \lambda_2 + \alpha_2) D(k_1) + \alpha_2\Big] A_1 \\ \qquad + \Big[(-ck_2 + \lambda_2 + \alpha_2) D(k_2) + \alpha_2\Big] A_2 = -\lambda_2 e^{-\frac{1}{\mu_2} y} \end{cases}$$

By injecting $D(k_i)$ in the above equation, we get

$$\begin{cases} \dfrac{1}{\mu_1 k_1 + 1} A_1 + \dfrac{1}{\mu_1 k_2 + 1} A_2 = e^{-\frac{1}{\mu_1} y} \\ -\dfrac{D(k_1)}{\mu_2 k_1 + 1} A_1 - \dfrac{D(k_2)}{\mu_2 k_2 + 1} A_2 = e^{-\frac{1}{\mu_2} y} \end{cases} \tag{14}$$

We get now the following theorem.

Theorem 2 *If* $\frac{\alpha_2}{c\mu_2}\varrho_1 + \frac{\alpha_1}{c\mu_1}\varrho_2 > 0$, *then it holds*

$$\begin{cases} \psi_1(u,y) = A_1 e^{k_1 u} + A_2 e^{k_2 u} \\ \psi_2(u,y) = -D(k_1) A_1 e^{k_1 u} - D(k_2) A_2 e^{k_2 u} \end{cases}$$

where k_1 *and* k_2 *are the two negative roots of (12), the constants* $D(k_i)$ *are defined by (13) and* $(A_1, A_2\ )$ *is the unique solution of the system of linear equations (14).*

If we assume that $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$, we obtain the classical risk model of Lundberg and Cramer (c.f. Grandell (1991)) where the claims arrive according to a Poisson process with parameter $\lambda > 0$.

Corollary 1 *if* $\lambda_1 = \lambda_2 = \lambda$ *and* $\mu_1 = \mu_2 = \mu$, *we obtain for the classical risk model with exponentially distributed claim amounts*

$$\psi(u, y) = \frac{\lambda\mu}{c} e^{-\varrho u - \frac{y}{\mu}}. \tag{15}$$

Proof $k_2 = -\varrho$ and k_1 is the negative root of

$$\mathcal{P}(k) = k^2 + k(\varrho - \frac{\alpha_1 + \alpha_2}{c}) - \frac{\alpha_1 + \alpha_2}{c\mu} = 0$$

Therefore $D(k_1) = \frac{\alpha 2}{\alpha_1}$, $D(k_2) = -1$ and the solutions of (14) are $A_1 = 0$ and $A_2 = \frac{\lambda\mu}{c} e^{-\frac{y}{\mu}}$. ■

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Abstract

In this paper we are concerned with the severity of ruin in Markov-modulated risk models. It is shown that the severity of ruin satisfy a system of differential equations. Explicit formula is derived for the severity of ruin in the particular case where one has two possible states for the process of environment and where the amounts of claims are distributed according to an exponential distribution.

Zusammenfassung

Dieser Artikel befasst sich mit der Schwere des Ruins in Markov-modulierten Risikomodellen. Es wird gezeigt, dass die Schwere des Ruins ein System von Differentialgleichungen erfüllt. Es werden explizite Formeln für die Schwere des Ruins hergeleitet für den Spezialfall, wo die Markov-Umgebung durch zwei mögliche Zustände definiert ist und die Schadenhöhen exponentiell verteilt sind.

Résumé

Ce papier traite de la sévérité de la ruine pour des risques dont les fréquences et les montants des sinistres dépendent d'un processus de Markov. On montre que la sévérité de la ruine satisfait un sytème d'équations différentielles. On dérive une formule explicite pour la sévérité de la ruine dans le cas particulier d'un processus de Markov à deux états où les montants des sinistres suivent une distribution exponentielle.