Risk measures related to the surplus process in the compound Markov binomial model

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1 Introduction

In Cossette et al. (2003), a compound Markov binomial model is presented which is an extension to the compound binomial model proposed by Gerber (1988a,b). The compound binomial model was also examined, among others, by Shiu (1989), Michel (1989), Willmot (1993), Dickson (1994), Dickson et al. (1995) and DeVylder and Marceau (1996). In the compound binomial model, the claim occurrence process is supposed independent whereas, in the compound Markov binomial model, time dependence is introduced in the claim occurrence process. In Cossette et al. (2003), we study the aggregate claim amount process and the computation of the ruin probabilities in the framework of this extension. Upper bounds and an asymptotic expression for the infinite-time ruin probability are also provided in Cossette et al. (2004). In this paper, we pursue our study of the compound Markov binomial model with the investigation of key risk measures related to the surplus process such as the distributions of the severity of ruin, the surplus one period prior to ruin and the claim causing ruin.
The compound Markov binomial model is a discrete-time risk model within which the surplus process $\{U_k, k \in \mathbb{N}\}$ is defined as

$$U_k = u + \sum_{j=1}^{k} (c - X_j) ,$$

for $k \in \mathbb{N}^+$, where $U_0 = u$ $(u \in \mathbb{N})$ corresponds to the initial surplus, c is the premium rate per period and X_j is the eventual claim amount in period j $(j \in \mathbb{N}^+)$. The premium rate c is assumed to be equal to 1. We suppose that at most one claim can occur per period. Therefore, the r.v.'s X_j are defined as

$$X_j = \begin{cases} B_j , & I_j = 1 \\ 0 , & I_j = 0 \end{cases} . \tag{1}$$

[1)]This research was funded by individual operating grants from the Natural Sciences and Engineering Research Council of Canada, by the Fonds de recherche sur la nature et les technologies du Gouvernement du Québec, and by a joint grant from the Chaire en Assurance L'Industrielle-Alliance (Université Laval).

We assume that the claim occurrence r.v.'s $\{I_k, k \in \mathbb{N}\}$ are no longer independent as in the compound binomial model. The dependence structure between the claim occurrence r.v.'s $\{I_k, k \in \mathbb{N}\}$ is introduced via a stationary homogeneous Markov chain with state space $\{0, 1\}$ and transition probability matrix

$$\underline{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}. \tag{2}$$

Throughout the paper, we denote the transition probabilities by p_{ij} meaning the probability of moving from state occurrence i to state occurrence j in a time period. We assume a correlation parameter π $(0 \leq \pi < 1)$ and a stationary claim occurrence probability q $(0 < q < 1)$. The transition probabilities $p_{ij} = \Pr(I_k = j \mid I_{k-1} = i)$ of (2) defined in terms of those two parameters are

$$\underline{P} = \begin{pmatrix} (1-q) + \pi q & q - \pi q \\ (1-q) - \pi(1-q) & q + \pi(1-q) \end{pmatrix}. \tag{3}$$

The initial probabilities are $\Pr(I_0 = 1) = q = 1 - \Pr(I_0 = 0)$. We can show that $\Pr(I_k = 1) = q$ for $k \in \mathbb{N}^+$, which means that the sequence is stationary.
In addition, we assume that the sequences $\{I_k, k \in \mathbb{N}\}$ and $\{B_k, k \in \mathbb{N}^+\}$ are mutually independent with $\{B_k, k \in \mathbb{N}^+\}$ being a sequence of i.i.d. r.v.'s with support $\mathbb{N}^+$, common probability mass function (p.m.f.) f_B and mean μ_B.
To ensure that the infinite-time ruin probability goes to 0 as $u \to \infty$, the parameters in the compound Markov binomial model are fixed such that

$$q\mu_B < 1. \tag{4}$$

In fact, the premium rate can be expressed as $1 = (1+\eta)q\mu_B$ where η is the (strictly positive) relative risk margin.
The paper is structured as follows: first, we briefly recall key results on the infinite time ruin probabilities in the compound Markov binomial model presented in Cossette et al. (2003). In Sections 2 and 3 respectively, the (defective) distribution of the severity of ruin and the (defective) distribution of the surplus one period prior ruin are derived from recursive algorithms. We study at the end of Section 3 the joint (defective) distribution of the surplus at ruin and the surplus one period prior ruin. In addition, the moments of the severity of ruin (defective) distribution are provided in Section 2. A similar study is made in Section 4 on the claim causing ruin. Throughout the paper, two specific claim amount distributions are considered since closed-form expressions can be obtained for the various risk measures studied. These specific claim amount distributions are also considered

in Cossette et al. (2004) to find an explicit expression for the ruin probability in the compound Markov binomial model. Finally, a numerical example is provided to illustrate and comment the risk measures discussed on a theoretical basis in previous sections.

2 Ruin probabilities

Let T denote the time of ruin associated to the surplus process U_k

$$T = \begin{cases} \inf_{k\in\mathbb{N}^+} \{k, U_k < 0\}, & \text{if } U_k \text{ falls below 0 at least once} \\ \infty, & \text{if } U_k \text{ never goes below 0} \end{cases} .$$

The conditional and unconditional infinite-time ruin probabilities are respectively defined as $\psi(u \mid i) = P(T < \infty \mid I_0 = i)$ for $i = 0, 1$ and $\psi(u) = P(T < \infty)$. Their respective complements are denoted $\phi(u \mid i)$ and $\phi(u)$ and are the conditional and unconditional infinite-time non-ruin probabilities

$$\begin{aligned} \phi(u \mid i) &= 1 - \psi(u \mid i) \\ &= \Pr(U_k \geq 0, \forall k \in \mathbb{N}^+ \mid I_0 = i), \end{aligned}$$

and

$$\begin{aligned} \phi(u) &= 1 - \psi(u) \\ &= \Pr(U_k \geq 0, \forall k \in \mathbb{N}^+). \end{aligned}$$

Obviously, the unconditional non-ruin probabilities can be calculated in terms of the conditional non-ruin probabilities with

$$\phi(u) = (1 - q)\phi(u \mid 0) + q\phi(u \mid 1). \tag{5}$$

In Cossette et al. (2003), a recursive algorithm is provided to compute the conditional infinite-time non-ruin probabilities in the compound Markov binomial model. We recall the result here but we omit the proof.
In the compound Markov binomial model, the infinite-time non-ruin probabilities are recursively obtained with

$$\phi(u \mid 0) = \frac{\phi(u - 1 \mid 0) - p_{01} \sum_{j=1}^{u} \phi(u - j \mid 1) f_B(j)}{p_{00}}, \tag{6}$$

and

$$\phi(u \mid 1) = \frac{p_{10}\phi(u \mid 0) + \pi \sum_{j=2}^{u+1} \phi(u+1-j \mid 1) f_B(j)}{p_{00} - \pi f_B(1)}, \tag{7}$$

for $u \in \mathbb{N}^+$. The starting points of these recursive formulas are

$$\phi(0 \mid 0) = \frac{1 - q\mu_B}{1 - q}, \tag{8}$$

and

$$\phi(0 \mid 1) = \frac{p_{10}}{p_{00} - \pi f_B(1)} \phi(0 \mid 0). \tag{9}$$

Alternative expressions for (6) and (7) are respectively

$$\phi(u \mid 0) = \phi(0 \mid 0) + \frac{q}{1-q} \sum_{j=0}^{u-1} \phi(j \mid 1)(1 - F_B(u-j)), \tag{10}$$

and

$$\phi(u \mid 1) = \phi(0 \mid 1) + \sum_{j=0}^{u-1} \frac{p_{01}(1 - F_B(u-j)) + \pi f_B(u+1-j)}{p_{00} - \pi f_B(1)} \phi(j \mid 1). \tag{11}$$

For specific claim amount distributions, explicit expressions for the ruin probabilities are given in Cossette et al. (2004).
It is obvious that ruin probability is a key indicator of the riskiness of a surplus process. However, other risk measures can help characterize the behavior of the surplus process and consequently improve our knowledge of it. This is why, in the following sections, the distribution of the severity of ruin, the surplus one period prior ruin and the distribution of the claim causing ruin, among others, are studied in the framework of the compound Markov binomial model.

3 Distribution of the severity of ruin

The distribution of the severity of ruin, also called the *distribution of the surplus at ruin*, was first studied by Gerber et al. (1987) within the classical compound

Poisson risk model and subsequently by others (e.g. Dufresne and Gerber (1988) and Dickson (1989)). Dickson (1994) and Dickson et al. (1995) are among the ones who examine it in the framework of the compound binomial model. Here, a similar study is made but this time in the framework of the compound Markov binomial model.

3.1 *Recursive formulas*

In this section, we propose, as in Dickson et al. (1995) within the compound binomial model, recursive algorithms to evaluate the (defective) distribution of the severity of ruin in the infinite-time horizon. In order to do so, we define the conditional and unconditional probability that ruin occurs and that the severity of ruin is not greater than $y \in \mathbb{N}^+$, denoted respectively by $G(u, y \mid i)$ and $G(u, y)$, as

$$G(u, y \mid i) = \Pr(T < \infty, U_T \geq -y \mid I_0 = i, U_0 = u)$$

and

$$G(u, y) = \Pr(T < \infty, U_T \geq -y \mid U_0 = u)\,,$$

for $u \in \mathbb{N}$ and $i \in \{0, 1\}$. According to our definitions, there is ruin if the surplus is strictly inferior to 0 which implies that $G(u, 0) = G(u, 0 \mid 0) = G(u, 0 \mid 1) = 0$. We have

$$G(u, y) = (1 - q)G(u, y \mid 0) + qG(u, y \mid 1)\,.$$

Note that $\lim_{y \to \infty} G(u, y \mid i) = \psi(u \mid i)$ and $\lim_{y \to \infty} G(u, y) = \psi(u)$. We also define $g(u, y \mid i)$ as

$$\begin{aligned} g(u, y \mid i) &= \Pr(T < \infty, U_T = -y \mid I_0 = i, U_0 = u) \\ &= G(u, y \mid i) - G(u, y - 1 \mid i) \end{aligned} \tag{12}$$

for $y \in \mathbb{N}^+$.

A first algorithm to obtain the conditional probability $G(u, y \mid i)$ for $u \in \mathbb{N}$, $y \in \mathbb{N}^+$ and $i \in \{0, 1\}$ is given in the following proposition.

Proposition 1 *In the compound Markov binomial model, the conditional probability that ruin occurs and that the severity of ruin is not greater than* $y \in \mathbb{N}^+$ *can be computed recursively with*

- *for $u = 0$ and $y \in \mathbb{N}^+$,*

$$G(0, y \mid 0) = \frac{q}{1-q}(E[B \wedge (y+1)] - 1)\,, \tag{13}$$

and

$$G(0, y \mid 1) = \frac{p_{10} G(0, y \mid 0) + \pi(F_B(y+1) - F_B(1))}{p_{00} - \pi f_B(1)}\,, \tag{14}$$

where $E[B \wedge y] = \sum_{x=1}^{\infty} \min(x, y) f_B(x)$.

- *for $u, y \in \mathbb{N}^+$,*

$$G(u, y \mid 0) = \frac{G(u-1, y \mid 0)}{p_{00}} - \frac{p_{01}\left(\sum_{k=1}^{u} G(u-k, y \mid 1) f_B(k) + \sum_{k=u+1}^{u+y} f_B(k)\right)}{p_{00}} \tag{15}$$

and

$$G(u, y \mid 1) = \frac{p_{10} G(u, y \mid 0)}{p_{00} - \pi f_B(1)} + \pi \frac{\sum_{k=2}^{u+1} G(u+1-k, y \mid 1) f_B(k) + \sum_{k=u+2}^{u+y+1} f_B(k)}{p_{00} - \pi f_B(1)}\,. \tag{16}$$

Proof: First, by conditioning on the claim occurrence and claim amount (if necessary) r.v.'s in the first period (i.e. I_1 and B_1) and given the stationarity of the surplus process, one finds

$$G(j, y \mid 0) = p_{00} G(j+1, y \mid 0) + p_{01} \sum_{k=1}^{j+1} G(j+1-k, y \mid 1) f_B(k) + p_{01} \sum_{k=j+2}^{j+y+1} f_B(k) \tag{17}$$

and

$$G(j,y \mid 1) = p_{10}G(j+1,y \mid 0) + p_{11}\sum_{k=1}^{j+1} G(j+1-k,y \mid 1)f_B(k) + p_{11}\sum_{k=j+2}^{j+y+1} f_B(k) \tag{18}$$

for $j \in \mathbb{N}$. The recursive formula (15) follows easily from (17). Also, substituting (17) in (18) yields

$$G(j,y \mid 1) = \frac{p_{10}G(j,y \mid 0)}{p_{00} - \pi f_B(1)} + \pi \frac{\sum_{k=2}^{j+1} G(j+1-k,y \mid 1)f_B(k) + \sum_{k=j+2}^{j+y+1} f_B(k)}{p_{00} - \pi f_B(1)}$$

which corresponds to (16). In addition, we derive (14) by combining both (17) and (18) at $j = 0$. The starting point of the recursive formula which is given in (13) remains to be proven. For that purpose, we first rearrange (17) and (18) as

$$G(j,y \mid 0) - G(j+1,y \mid 0) = p_{01}\left(\sum_{k=1}^{j+1} G(j+1-k,y \mid 1)f_B(k) + \sum_{k=j+2}^{j+y+1} f_B(k) - G(j+1,y \mid 0)\right) \tag{19}$$

and

$$G(j,y \mid 1) - \sum_{k=1}^{j+1} G(j+1-k,y \mid 1)f_B(k) - \sum_{k=j+2}^{j+y+1} f_B(k) = p_{10}\left(G(j+1,y \mid 0) - \sum_{k=1}^{j+1} G(j+1-k,y \mid 1)f_B(k) - \sum_{k=j+2}^{j+y+1} f_B(k)\right), \tag{20}$$

given the equality $p_{i0} = 1 - p_{i1}$ for $i \in \{0,1\}$. Dividing (19) by (20) and summing for $j = 0, \dots, u-1$ yields

$$\begin{aligned}
&\sum_{j=0}^{u-1}(G(j,y \mid 0) - G(j+1,y \mid 0)) \\
&\quad = \frac{p_{01}}{p_{10}}\left(\sum_{j=0}^{u-1}\sum_{k=j+2}^{j+y+1} f_B(k) - \sum_{j=0}^{u-1} G(j,y \mid 1)\right. \\
&\qquad \left. + \sum_{j=0}^{u-1}\sum_{k=1}^{j+1} G(j+1-k,y \mid 1) f_B(k)\right) \\
&\quad = \frac{p_{01}}{p_{10}}\left(\sum_{j=0}^{u-1}\sum_{k=j+2}^{j+y+1} f_B(k) - \sum_{j=0}^{u-1} G(j,y \mid 1)(1-F_B(u-j))\right). \qquad (21)
\end{aligned}$$

Taking the limit for $u \to \infty$ on both sides of (21) leads to

$$\begin{aligned}
&G(0,y \mid 0) \\
&\quad = \frac{p_{01}}{p_{10}}\left(\sum_{j=0}^{\infty}\sum_{k=j+2}^{j+y+1} f_B(k) - \lim_{u\to\infty}\sum_{j=0}^{u-1} G(j,y \mid 1)(1-F_B(u-j))\right) \\
&\quad = \frac{p_{01}}{p_{10}}\left(\sum_{j=0}^{\infty}\sum_{k=j+2}^{j+y+1} f_B(k) - \lim_{u\to\infty}\sum_{j=1}^{u} G(u-j,y \mid 1)(1-F_B(j))\right) \\
&\quad = \frac{p_{01}}{p_{10}}\left(\sum_{j=0}^{\infty}\sum_{k=j+2}^{j+y+1} f_B(k)\right. \\
&\qquad \left. - \lim_{u\to\infty}\sum_{j=0}^{\infty} G(u-j,y \mid 1)(1-F_B(j))1_{j\in\{0,\dots,u-1\}}\right). \qquad (22)
\end{aligned}$$

In order to apply the dominated convergence theorem to (22), we need a summable function which majorates $G(j,y \mid 1)(1-F_B(u-j))1_{j\in\{0,\dots,u-1\}}$ for $\forall j \in \mathbb{N}$. Such a function can be

$$\left|G(u-j-1,y \mid 1)(1-F_B(j+1))1_{j\in\{0,\dots,u-1\}}\right| \leq (1-F_B(j+1))$$

for which it follows that $\sum_{j=0}^{\infty}(1-F_B(j+1)) = \mu_B - 1 < \infty$ from (4).

Consequently, (22) becomes

$$
\begin{aligned}
G(0,y\mid 0) &= \frac{p_{01}}{p_{10}}\sum_{j=0}^{\infty}\sum_{k=j+2}^{j+y+1} f_B(k) \\
&\quad - \frac{p_{01}}{p_{10}}\sum_{j=1}^{\infty}(1-F_B(j))\lim_{u\to\infty} G(u-j,y\mid 1)1_{j\in\{0,\dots,u-1\}} \\
&= \frac{q}{1-q}\sum_{j=0}^{\infty}\sum_{k=2}^{y+1} f_B(j+k) \\
&= \frac{q}{1-q}\left(\sum_{j=0}^{y}(1-F_B(j))-1\right) \\
&= \frac{q}{1-q}(E[B\wedge(y+1)]-1), \qquad (23)
\end{aligned}
$$

for $y\in\mathbb{N}^+$. □

From (12), one easily obtains $g(u,y\mid i)$ and, in particular, assuming an initial surplus u equal to 0, we have

$$
g(0,y\mid i)=\begin{cases}\dfrac{q}{1-q}(1-F_B(y)), & i=0\\[2ex] \dfrac{p_{01}(1-F_B(y))+\pi f_B(y+1)}{p_{00}-\pi f_B(1)}, & i=1\end{cases}. \qquad (24)
$$

When $\pi=0$, we find the same expression of the probability that ruin occurs and that the severity of ruin is y as the one in Dickson et al. (1995).

However, the drawback of the algorithm presented in Proposition 1 is its unstability in the sense of Panjer and Wang (1993) which is why we present an alternative equivalent but stable algorithm in Proposition 2.

Proposition 2 *In the compound Markov binomial model, a stable algorithm to compute the conditional probabilities* $G(u,y\mid i)$ *for* $u,y\in\mathbb{N}^+$ *and* $i\in\{0,1\}$ *is*

$$
G(u,y\mid i)=\sum_{j=1}^{u}G(u-j,y\mid 1)g(0,j\mid i)+\sum_{j=u+1}^{u+y}g(0,j\mid i) \qquad (25)
$$

where $g(0,j\mid i)$ *and the starting points of the recursive formula,* $G(0,y\mid 0)$ *and* $G(0,y\mid 1)$*, are as given respectively in* (24), (13) *and* (14).

Proof: We first rearrange (21) as

$$\begin{aligned} G(u,y \mid 0) &= G(0,y \mid 0) \\ &\quad + \frac{q}{1-q}\Bigg(\sum_{j=0}^{u-1} G(j,y \mid 1)(1-F_B(u-j)) - \sum_{j=0}^{u-1}\sum_{k=j+2}^{j+y+1} f_B(k)\Bigg) \\ &= G(0,y \mid 0) \\ &\quad + \frac{q}{1-q}\Bigg(\sum_{j=1}^{u} G(u-j,y \mid 1)(1-F_B(j)) - \sum_{k=1}^{y}\sum_{j=1}^{u} f_B(k+j)\Bigg). \end{aligned} \tag{26}$$

From (23) and (24), (26) becomes

$$\begin{aligned} G(u,y \mid 0) &= \sum_{j=1}^{u} G(u-j,y \mid 1) g(0,j \mid 0) \\ &\quad + \frac{q}{1-q}\Bigg((E[B \wedge (y+1)] - 1) - \sum_{k=1}^{y}\sum_{j=1}^{u} f_B(k+j)\Bigg) \\ &= \sum_{j=1}^{u} G(u-j,y \mid 1) g(0,j \mid 0) \\ &\quad + \frac{q}{1-q}\sum_{j=1}^{y}(1-F_B(u+j)) \\ &= \sum_{j=1}^{u} G(u-j,y \mid 1) g(0,j \mid 0) \\ &\quad + \frac{q}{1-q}\sum_{j=u+1}^{u+y}(1-F_B(j)) \\ &= \sum_{j=1}^{u} G(u-j,y \mid 1) g(0,j \mid 0) \\ &\quad + \sum_{j=u+1}^{u+y} g(0,j \mid 0), \end{aligned} \tag{27}$$

which corresponds to (25) for $i = 0$.

To prove (25) for $i = 1$, we first combine (16) and (27)

$$G(u, y \mid 1) = p_{10} \frac{\sum_{k=1}^{u} G(u-k, y \mid 1) g(0, k \mid 0) + \sum_{k=u+1}^{u+y} g(0, k \mid 0)}{p_{00} - \pi f_B(1)}$$
$$+ \pi \frac{\sum_{k=2}^{u+1} G(u+1-k, y \mid 1) f_B(k) + \sum_{k=u+2}^{u+y+1} f_B(k)}{p_{00} - \pi f_B(1)}$$
$$= \sum_{k=1}^{u} \frac{p_{10} g(0, k \mid 0) + \pi f_B(k+1)}{p_{00} - \pi f_B(1)} G(u-k, y \mid 1)$$
$$+ \sum_{k=u+1}^{u+y} \frac{p_{10} g(0, k \mid 0) + \pi f_B(k+1)}{p_{00} - \pi f_B(1)}$$

which becomes, using (24),

$$G(u, y \mid 1) = \sum_{j=1}^{u} G(u-j, y \mid 1) g(0, j \mid 1) + \sum_{j=u+1}^{u+y} g(0, j \mid 1)$$

which completes the proof. □

Relation (25) for both $i = 0, 1$ have a nice interpretation. The conditional probability that ruin occurs and that the severity of that ruin is not greater than y knowing that the claim occurrence process is in state i ($i \in \{0, 1\}$) at time 0 can be interpreted as follows:

a) the first term of (25) gives, for a fixed value of j ($j \in \{1, 2, \ldots, u\}$), the probability that the surplus falls below its initial surplus for the first time to a new level $u - j$ given $I_0 = i$ and that eventually ruin occurs and that severity of ruin is not greater than y.

b) the second term of (25) gives the probability that the first time the surplus falls below its initial level, the drop in the surplus according to its starting level u is $\{u+1, \ldots, u+y\}$ given $I_0 = i$ which ensures that ruin occurs.

For the remainder of this subsection, we consider two claim amount distributions which admit, as shown in Cossette et al. (2004), an explicit expression for the ruin probabilities in the compound Markov binomial model (i.e. $B \in \{1, 2\}$ and B has a geometric distribution). As we will see, these two specific distributions also lead to a closed-form expression for the conditional probability that ruin occurs and that the severity of ruin is not greater than $y \in \mathbb{N}^+$.

***Example 1** Assume that the claim amount r.v. B takes values in $\{1,2\}$ with $f_B(2) > 0$. Therefore, if a claim occurs, the surplus could stay the same or decrease from one unit of its previous level. Otherwise, the surplus always increases from one unit compared to its previous level. In such a case, a closed-form expression for $G(u, y \mid i)$ for $u \in \mathbb{N}$, $y \in \mathbb{N}^+$ and $i \in \{0,1\}$ is given by*

$$G(u, y \mid i) = G(0, y \mid i)\left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)}\right)^u . \tag{28}$$

Proof: Since $B \in \{1,2\}$, (24) for $i = 1$ can be written as

$$g(0, y \mid 1) = \begin{cases} \dfrac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)}, & y = 1 \\ 0, & y \neq 1 \end{cases}, \tag{29}$$

which implies that (25) at $i = 1$ becomes

$$G(u, y \mid 1) = \frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} G(u-1, y \mid 1), \tag{30}$$

for $u \in \mathbb{N}^+$. Recursive applications of (30) leads to

$$G(u, y \mid 1) = \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)}\right)^u G(0, y \mid 1), \tag{31}$$

for $u \in \mathbb{N}^+$ which proves (28) for $i = 1$.
Similarly, when $B \in \{1,2\}$, (24) at $i = 0$ can be simplified to

$$g(0, y \mid 0) = \begin{cases} \dfrac{q}{1-q} f_B(2), & y = 1 \\ 0, & y \neq 1 \end{cases}. \tag{32}$$

From (32) and (31), (25) at $i = 0$ becomes

$$\begin{aligned} G(u, y \mid 0) &= \frac{q}{1-q} f_B(2) G(u-1, y \mid 1) \\ &= \frac{q}{1-q} f_B(2)\left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)}\right)^{u-1} G(0, y \mid 1) \end{aligned} \tag{33}$$

for $u \in \mathbb{N}^+$. Since (13) and (14) are respectively given by

$$G(0, y \mid 0) = \frac{q}{1-q} f_B(2),$$

and

$$G(0, y \mid 1) = \frac{p_{11}}{p_{00} - \pi f_B(1)} f_B(2)$$

when $B \in \{1, 2\}$, (33) can be rewritten as

$$G(u, y \mid 0) = \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)}\right)^u G(0, y \mid 0) . \qquad \square$$

Example 2 *In cases where the claim amount r.v. B has a geometric distribution with p.m.f. $f_B(i) = (1-\alpha)\alpha^{i-1}$ for $i \in \mathbb{N}^+$, an explicit expression for $G(u, y \mid i)$ for $u \in \mathbb{N}$, $y \in \mathbb{N}^+$ and $i \in \{0, 1\}$ is given by*

$$G(u, y \mid i) = G(0, y \mid i)\left(\frac{\alpha}{p_{00} - \pi(1-\alpha)}\right)^u . \tag{34}$$

First, we highlight a result that will be helpful in the proof of (34). When B is geometrically distributed with parameter α, one can easily find that

$$g(0, y+1 \mid i) = \alpha g(0, y \mid i) , \tag{35}$$

for $y \in \mathbb{N}^+$.

Proof of (34): Subtracting $\alpha G(u, y \mid 1)$ from $G(u+1, y \mid 1)$ and, given (25) and (35) both at $i = 1$, one deduces that

$$\begin{aligned} &G(u+1, y \mid 1) - \alpha G(u, y \mid 1) \\ &\quad = \sum_{j=1}^{u+1} G(u+1-j, y \mid 1) g(0, j \mid 1) + \sum_{j=u+2}^{u+y+1} g(0, j \mid 1) \\ &\quad\quad - \alpha\left(\sum_{j=1}^{u} G(u-j, y \mid 1) g(0, j \mid 1) + \sum_{j=u+1}^{u+y} g(0, j \mid 1)\right) \\ &\quad = g(0, 1 \mid 1) G(u, y \mid 1) \end{aligned}$$

which can be rearranged as

$$G(u+1, y \mid 1) = \frac{\alpha}{p_{00} - \pi f_B(1)} G(u, y \mid 1) \tag{36}$$

for $u \in \mathbb{N}$ and $y \in \mathbb{N}^+$. Recursive applications of (36) yields

$$G(j, y \mid 1) = \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^j G(0, y \mid 1) , \tag{37}$$

for $j \in \mathbb{N}$ and $y \in \mathbb{N}^+$ which completes the proof of (34) for $i = 1$.
Moreover, for the geometric distribution, it follows from (25) at both $i = 0$ and 1, when combined to (24), that

$$G(j, y \mid 1) = \frac{1-q}{q} \cdot \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)} G(j, y \mid 0) . \tag{38}$$

Multiplying $G(j, y \mid 0)$ by α and then subtracting it from $G(j+1, y \mid 0)$ yields

$$G(j+1, y \mid 0) - \alpha G(j, y \mid 0) = g(0, 1 \mid 0) G(j, y \mid 1)$$

which becomes, from (38),

$$\begin{aligned} G(j+1, y \mid 0) &= \left(\alpha + \frac{1-q}{q} \cdot \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)} g(0, 1 \mid 0)\right) G(j, y \mid 0) \\ &= \left(\alpha + \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)} \alpha\right) G(j, y \mid 0) \\ &= \frac{\alpha}{p_{00} - \pi(1-\alpha)} G(j, y \mid 0) , \end{aligned} \tag{39}$$

for $j \in \mathbb{N}$ and $y \in \mathbb{N}^+$. Recursive applications of (39) result in (34) for $i = 0$. □

3.2 *Link to the ruin probabilities*

Based on probabilistic arguments, we can find (similarly to Dickson (1994) and Dickson et al. (1995) in the compound binomial model) the following stable alternative recursive formulas to (6) and (7)

$$\phi(u \mid i) = \phi(0 \mid i) + \sum_{j=1}^{u} g(0, j \mid i) \phi(u - j \mid 1) . \tag{40}$$

The proofs of (40) for $i = 0, 1$ are direct consequences of the combinations of (24) with (10) and (11).
The expression of the conditional non-ruin probabilities $\phi(u \mid i)$ given in (40) can be interpreted as

a) the first term corresponds to the probability that the surplus process never falls below their initial surplus u given $I_0 = i$.

b) the second term gives, for a fixed value of $j \in \{1, \ldots, u\}$, the probability that the surplus falls below its initial surplus for the first time to a new level of $u - j$ given $I_0 = i$ and that eventually ruin does not occur.

The conditional non-ruin probabilities $\phi(0 \mid 0)$ and $\phi(0 \mid 1)$, given in Cossette et al. (2003), could also be found using both (13) and (14) since

$$\begin{aligned}
\phi(0 \mid 0) &= 1 - \psi(0 \mid 0) \\
&= 1 - \lim_{y \to \infty} G(0, y \mid 0) \\
&= 1 - \frac{q}{1-q}(\mu_B - 1) \\
&= \frac{1 - q\mu_B}{1-q},
\end{aligned}$$

and

$$\begin{aligned}
\phi(0 \mid 1) &= 1 - \psi(0 \mid 1) \\
&= 1 - \lim_{y \to \infty} G(0, y \mid 1) \\
&= 1 - \frac{p_{01}(\mu_B - 1) + \pi(1 - f_B(1))}{p_{00} - \pi f_B(1)} \\
&= \frac{p_{10}}{p_{00} - \pi f_B(1)} \phi(0 \mid 0) .
\end{aligned}$$

3.3 *Moments of the ruin severity*

Based on Dickson et al. (1995) within the compound binomial model, we derive the moments of the ruin severity in the compound Markov binomial model. From a practical point of view, the conditional moments of the ruin severity given that ruin actually occurs are more interesting. Here, we find the unconditional moments since the conditional ones can be obtained easily by just dividing them by the probability that ruin occurs.

Let us denote by W the amount of surplus at ruin and assume that W takes value 0 if no ruin occurs

$$W = \begin{cases} U_T , & \text{if } T < \infty \\ 0 , & \text{if } T = \infty \end{cases} .$$

The k^{th} conditional moment of the ruin severity given that $U_0 = u$ and $I_0 = i$ is denoted $E[W^k \mid u, i]$ and defined as

$$\begin{aligned} E[W^k \mid u, i] &= E[W^k \mid U_0 = u, I_0 = i] \\ &= \sum_{w=1}^{\infty} w^k g(u, w \mid i) . \end{aligned} \tag{41}$$

In particular, when $u = 0$, (41) for $i = 0$ and $i = 1$ can be rewritten respectively as

$$E[W^k \mid 0, 0] = \frac{q}{1-q} \sum_{w=1}^{\infty} w^k (1 - F_B(w)) , \tag{42}$$

and

$$\begin{aligned} E[W^k \mid 0, 1] &= \sum_{w=1}^{\infty} w^k \frac{p_{01}(1 - F_B(w)) + \pi f_B(w+1)}{p_{00} - \pi f_B(1)} \\ &= \frac{p_{10}}{p_{00} - \pi f_B(1)} E[W^k \mid 0, 0] \\ &\quad + \frac{\pi}{p_{00} - \pi f_B(1)} \sum_{w=1}^{\infty} w^k f_B(w+1) . \end{aligned} \tag{43}$$

Let us consider the first three moments of (42) and (43). It can be proven that the first three conditional moments of the severity of ruin (when $I_0 = 0$) are given by

$$\begin{aligned} E[W \mid 0, 0] &= \frac{q}{1-q} \cdot \frac{1}{2} (E[B^2] - \mu_B) , \\ E[W^2 \mid 0, 0] &= \frac{q}{1-q} \left(\frac{1}{3} E[B^3] - \frac{1}{2} E[B^2] + \frac{1}{6} \mu_B \right) \end{aligned}$$

and finally,

$$E[W^3 \mid 0, 0] = \frac{q}{1-q} \left(\frac{1}{4} E[B^4] - \frac{1}{2} E[B^3] + \frac{1}{4} E[B^2] \right) .$$

Similarly, for $I_0 = 1$, the first three conditional moments of the severity of ruin are given by

$$\begin{aligned} E[W \mid 0, 1] &= \frac{p_{10}}{p_{00} - \pi f_B(1)} E[W \mid 0, 0] + \frac{\pi}{p_{00} - \pi f_B(1)} (\mu_B - 1) , \\ E[W^2 \mid 0, 1] &= \frac{p_{10}}{p_{00} - \pi f_B(1)} E[W^2 \mid 0, 0] \\ &\quad + \frac{\pi}{p_{00} - \pi f_B(1)} (E[B^2] - 2\mu_B + 1) , \end{aligned}$$

and finally,

$$E[W^3 \mid 0,1] = \frac{p_{10}}{p_{00} - \pi f_B(1)} E[W^3 \mid 0,0] + \frac{\pi}{p_{00} - \pi f_B(1)} (E[B^3] - 3E[B^2] + 3\mu_B - 1) .$$

For $u \in \mathbb{N}$, one can compute recursively $E[W^k \mid u, i]$ with (41) and

$$g(u, y \mid i) = g(0, u + y \mid i) + \sum_{k=1}^{u} g(0, k \mid i) g(u - k, y \mid 1) .$$

See Dickson et al. (1995) for more details on the procedure.

4 Distribution of the surplus a period prior ruin

The distribution of the surplus immediately prior to ruin was first considered in the framework of the classical risk model by Dufresne and Gerber (1988) and further examined by Dickson (1992). In the compound binomial model, the distribution of the surplus one period prior ruin has been studied by Dickson et al. (1995). Since the compound Markov binomial model is an extension to the compound binomial model, the results on the distribution of the surplus one period prior ruin presented in this paper extends the ones provided in Dickson et al. (1995).

Given an initial surplus $u \in \mathbb{N}$, we denote the conditional and unconditional probabilities that ruin occurs and that the surplus a period prior ruin is inferior or equal to y respectively by

$$F(u, y \mid i) = \Pr(T < \infty, U_{T-1} \leq y \mid U_0 = u, I_0 = i) ,$$

and

$$F(u, y) = \Pr(T < \infty, U_{T-1} \leq y \mid U_0 = u) ,$$

for $i \in \{0, 1\}$ and $u, y \in \mathbb{N}$. It follows that

$$F(u, y) = (1 - q) F(u, y \mid 0) + q F(u, y \mid 1) .$$

We also define $f(u, y \mid i)$ as

$$f(u, y \mid i) = \begin{cases} F(u, y \mid i) - F(u, y - 1 \mid i) , & y \in \mathbb{N}^+ \\ F(u, 0 \mid i) , & y = 0 \end{cases} , \qquad (44)$$

for $i \in \{0, 1\}$ and $u \in \mathbb{N}$.

In the following proposition, we give a first recursive algorithm to obtain the conditional (defective) distribution of the surplus one period prior ruin. The proof here is similar to the one given in Proposition 1 and we therefore outline only the major steps.

Proposition 3 *In the compound Markov binomial model, the conditional (defective) distribution of the surplus a period prior ruin is given by*

- *for* $u = 0$ *and* $y \in \mathbb{N}$,

$$F(0, y \mid 0) = \frac{q}{1-q}\left(E[B \wedge (y+2)] - 1\right); \tag{45}$$

$$F(0, y \mid 1) = \frac{p_{10} F(0, y \mid 0) + \pi(1 - f_B(1))}{p_{00} - \pi f_B(1)}; \tag{46}$$

- *for* $u \in \mathbb{N}^+$ *and* $y \in \mathbb{N}$,

$$F(u, y \mid 0) = \frac{F(u-1, y \mid 0)}{p_{00}} - p_{01} \frac{\sum_{k=1}^{u} F(u-k, y \mid 1) f_B(k)}{p_{00}} - p_{01} \frac{1_{\{u \le y+1\}} \sum_{k=u+1}^{\infty} f_B(k)}{p_{00}}; \tag{47}$$

$$F(u, y \mid 1) = \frac{p_{10} F(u, y \mid 0)}{p_{00} - \pi f_B(1)} + \pi \frac{\sum_{j=1}^{u} F(u-j, y \mid 1) f_B(j+1)}{p_{00} - \pi f_B(1)} + \pi \frac{+1_{\{u \le y\}} \sum_{j=u+1}^{\infty} f_B(j+1)}{p_{00} - \pi f_B(1)}, \tag{48}$$

where, throughout this paper, we assume that $1_A = \begin{cases} 1, & \text{if } A \text{ is true} \\ 0, & \text{otherwise} \end{cases}$.

Proof: First, we condition $F(j, y \mid i)$ on both r.v.'s I_1 and B_1 (if necessary) and given the stationarity of the surplus process, one finds

$$F(j, y \mid i) = p_{i0} F(j+1, y \mid 0) + p_{i1} \left(\sum_{k=1}^{j+1} F(j+1-k, y \mid 1) f_B(k) + 1_{\{j \leq y\}} \sum_{k=j+2}^{\infty} f_B(k) \right) \tag{49}$$

for $j, y \in \mathbb{N}$ and $i \in \{0, 1\}$. One easily sees that (47) is derived from (49) at $i = 0$ and (48) is obtained by combining (49) at both $i = 0$ and $i = 1$. Moreover, (46) is a direct consequence of (49) at $i = 0$ and $j = 0$ and (49) at $i = 1$ and $j = 0$. However, one must still prove (45) which corresponds to the starting point of the recursive formulas.
For that purpose, we first sum (49) for $j = 0, 1, \ldots, u-1$ and given the equality $p_{i0} = 1 - p_{i1}$ for $i \in \{0, 1\}$, one finds

$$\begin{aligned} F(0, y \mid 0) - F(u, y \mid 0) &= \frac{p_{01}}{p_{10}} \sum_{j=0}^{u-1} 1_{\{j \leq y\}} \sum_{k=j+2}^{\infty} f_B(k) \\ &\quad - \frac{p_{01}}{p_{10}} \sum_{j=0}^{u-1} \left(F(j, y \mid 1) - \sum_{k=1}^{j+1} F(j+1-k, y \mid 1) f_B(k) \right) \\ &= \frac{p_{01}}{p_{10}} \sum_{j=0}^{u-1} \left(1_{\{j \leq y\}} \sum_{k=j+2}^{\infty} f_B(k) - F(j, y \mid 1)(1 - F_B(u-j)) \right), \end{aligned} \tag{50}$$

for $u \in \mathbb{N}^+$ and $y \in \mathbb{N}$. Then, taking the limit $u \to \infty$ and applying the dominated convergence theorem (as in Proposition 1), we obtain

$$\begin{aligned} F(0, y \mid 0) &= \frac{q}{1-q} \lim_{u \to \infty} \left(\sum_{j=0}^{u-1} 1_{\{j \leq y\}} \sum_{k=j+2}^{\infty} f_B(k) \right. \\ &\qquad \left. - \sum_{j=1}^{u} F(u-j, y \mid 1)(1 - F_B(j)) \right) \\ &= \frac{q}{1-q} \sum_{j=0}^{y} \sum_{k=j+2}^{\infty} f_B(k) \\ &= \frac{q}{1-q} (E[B \wedge (y+2)] - 1) \end{aligned} \tag{51}$$

since $\lim_{u \to \infty} F(u, y \mid 0) = 0$ for $y \in \mathbb{N}$ due to (4). □

From (44) and results of Proposition 3, one easily obtains $f(u, y \mid i)$. In particular, assuming an initial surplus u equal to 0, we have

$$f(0, y \mid i) = \begin{cases} \frac{q}{1-q}(1 - F_B(y+1)), & i = 0 \\ \frac{p_{01}}{p_{00} - \pi f_B(1)}(1 - F_B(y+1)), & i = 1 \end{cases}. \tag{52}$$

for $y \in \mathbb{N}$. When $\pi = 0$, (52) is equivalent to the associated result obtained in Dickson et al. (1995) in the framework of the compound binomial model.

However, the algorithm proposed in Proposition 3 is unstable according to Panjer and Wang (1993). We remedy to this weakness by presenting an alternative and stable algorithm in the proposition that follows.

Proposition 4 *In the compound Markov binomial model, a stable algorithm to compute the conditional (defective) distribution of the surplus one period prior ruin is*

$$F(u, y \mid 0) = \begin{cases} \sum_{j=1}^{u} g(0, j \mid 0) F(u-j, y \mid 1) + \sum_{j=u+1}^{y+1} g(0, j \mid 0), & u \leq y \\ \sum_{j=1}^{u} g(0, j \mid 0) F(u-j, y \mid 1), & u > y \end{cases} \tag{53}$$

and

$$F(u, y \mid 1) = \begin{cases} \sum_{j=1}^{u} g(0, j \mid 1) F(u-j, y \mid 1) + \sum_{j=u+1}^{y+1} g(0, j \mid 1) \\ \quad + \frac{\pi(1 - F_B(y+2))}{p_{00} - \pi f_B(1)}, & u \leq y \\ \sum_{j=1}^{u} g(0, j \mid 1) F(u-j, y \mid 1), & u > y \end{cases} \tag{54}$$

for $u \in \mathbb{N}^+$ and $y \in \mathbb{N}$ where $g(0, j \mid i)$ is as given in (24). The starting points of the algorithm, $F(0, y \mid 0)$ and $F(0, y \mid 1)$, are respectively given in (45) and (46).

Proof: We begin with the proof of (53). Using (51), we rearrange (50) as

$$F(u,y \mid 0) = \frac{q}{1-q}\Bigg(\sum_{j=0}^{y}\sum_{k=j+2}^{\infty} f_B(k) - \sum_{j=0}^{u-1} 1_{\{j\le y\}} \sum_{k=j+2}^{\infty} f_B(k)\Bigg)$$

$$+ \frac{q}{1-q}\sum_{j=0}^{u-1} F(j,y \mid 1)(1-F_B(u-j))$$

$$= \frac{q}{1-q}\sum_{j=1}^{u} F(u-j,y \mid 1)(1-F_B(j))$$

$$+ \frac{q}{1-q}\Bigg(\sum_{j=0}^{y}(1-F_B(j+1)) - \sum_{j=0}^{u-1} 1_{\{j\le y\}}(1-F_B(j+1))\Bigg)$$

$$= \frac{q}{1-q}\sum_{j=1}^{u} F(u-j,y \mid 1)(1-F_B(j))$$

$$+ 1_{\{u\le y\}}\sum_{j=u}^{y}\frac{q}{1-q}(1-F_B(j+1))\,. \quad (55)$$

From (24) at $i=0$, (55) can be rewritten as

$$F(u,y \mid 0) = \sum_{j=1}^{u} F(u-j,y \mid 1) g(0,j \mid 0) + 1_{\{u\le y\}} \sum_{j=u+1}^{y+1} g(0,j \mid 0), \quad (56)$$

which corresponds to (53).

To obtain (54), we modify (48) with (56) which yields

$$F(u,y \mid 1) = \sum_{j=1}^{u} \frac{p_{10} g(0,j \mid 0) + \pi f_B(j+1)}{p_{00} - \pi f_B(1)} F(u-j,y \mid 1)$$

$$+ 1_{\{u\le y\}} \frac{p_{10}\sum_{j=u+1}^{y+1} g(0,j \mid 0) + \pi \sum_{j=u+1}^{\infty} f_B(j+1)}{p_{00} - \pi f_B(1)}\,. \quad (57)$$

From (24) at $i = 1$, (57) becomes

$$F(u, y \mid 1) = \sum_{j=1}^{u} g(0, j \mid 1) F(u - j, y \mid 1) \\ + 1_{\{u \le y\}} \left(\sum_{j=u+1}^{y+1} g(0, j \mid 1) + \frac{\pi}{p_{00} - \pi f_B(1)} (1 - F_B(y + 2)) \right)$$

which completes the proof of (54). □

The stable algorithm provided in Proposition 4 is useful to obtain an explicit expression for the distribution of the surplus one period prior ruin for the two specific claim amount distributions introduced previously in the context of the severity of ruin.

Example 3 *(continuation of Example 1) For cases where $B \in \{1, 2\}$ with $f_B(2) > 0$, a closed-form expression for the conditional (defective) distribution of the surplus one period prior ruin is*

$$F(u, y \mid i) = \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} \right)^u F(0, y \mid i), \tag{58}$$

for $u \in \mathbb{N}^+$, $y \in \mathbb{N}$ and $i \in \{0, 1\}$.

Proof: From (29), (54) becomes

$$\begin{aligned} F(u, y \mid 1) &= g(0, 1 \mid 1) F(u - 1, y \mid 1) \\ &= \frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} F(u - 1, y \mid 1), \end{aligned} \tag{59}$$

for $u \in \mathbb{N}^+$ and $y \in \mathbb{N}$. Recursive applications of (59) yields (58) for $i = 1$. Similarly, by combining (32) and (53), one has

$$F(u, y \mid 0) = \frac{q f_B(2)}{1 - q} F(u - 1, y \mid 1), \tag{60}$$

for $u \in \mathbb{N}^+$ and $y \in \mathbb{N}$. From (58) at $i = 1$, (60) becomes

$$F(u, y \mid 0) = \frac{q f_B(2)}{1 - q} \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} \right)^{u-1} F(0, y \mid 1). \tag{61}$$

In the case where $B \in \{1, 2\}$, (45) and (46) can be simplified to $F(0, y \mid 0) = \frac{q}{1 - q} f_B(2)$ and $F(0, y \mid 1) = \frac{p_{11}}{p_{00} - \pi f_B(1)} f_B(2)$ which permits to rewrite

(61) as

$$F(u,y \mid 0) = \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)}\right)^u F(0,y \mid 0)\,. \qquad \square$$

Example 4 *(continuation of Example 2) Assuming a geometric distribution with parameter α for the claim amount r.v. B, the probability that ruin occurs and that the surplus a period prior ruin is inferior or equal to y is given by*

$$F(k,y \mid 0) = \begin{cases} \frac{q}{1-q} \cdot \frac{p_{00} - \pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^k \\ \cdot \Bigg(F(0,y \mid 1) - \alpha\pi (p_{00} - \pi f_B(1))^y \\ \quad - \alpha p_{01} \frac{(p_{00} - \pi f_B(1))^{y+1} - \alpha^{y+1}}{p_{00} - \pi f_B(1) - \alpha} \Bigg), & y < k \\ \\ \frac{q}{1-q} \cdot \frac{p_{00} - \pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} \Bigg(\left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^k F(0,y \mid 1) \\ - \frac{\pi}{p_{00} - \pi f_B(1)} \alpha^{y+2} \\ - \frac{p_{01}}{p_{00} - \pi f_B(1)} \alpha^{y+2} \frac{1 - \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^k}{1 - \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)} \Bigg), & y \geq k \end{cases} \tag{62}$$

and

$$F(k,y \mid 1) = \begin{cases} \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^k \Bigg(F(0,y \mid 1) - \alpha\pi (p_{00} - \pi f_B(1))^y \\ \quad - \alpha p_{01} \frac{(p_{00} - \pi f_B(1))^{y+1} - \alpha^{y+1}}{p_{00} - \pi f_B(1) - \alpha} \Bigg), & y < k \\ \\ \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^k F(0,y \mid 1) \\ - \frac{p_{01}}{p_{00} - \pi f_B(1)} \alpha^{y+2} \frac{1 - \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)^k}{1 - \left(\frac{\alpha}{p_{00} - \pi f_B(1)}\right)}, & y \geq k \end{cases} \tag{63}$$

for $k \in \mathbb{N}^+$.

Proof: Subtracting $\alpha F(u,y \mid 1)$ from $F(u+1,y \mid 1)$ and, given (54) and (35) both at $i=1$, one deduces that

$$\begin{aligned}
&F(u+1,y \mid 1) - \alpha F(u,y \mid 1) \\
&= g(0,1 \mid 1)F(u,y \mid 1) \\
&\quad - 1_{\{u=y\}} \frac{\pi}{p_{00} - \pi f_B(1)}(1 - F_B(y+2)) \\
&\quad - 1_{\{u \le y\}} \left(g(0,y+2 \mid 1) - \frac{\pi(1-\alpha)}{p_{00} - \pi f_B(1)}(1 - F_B(y+2)) \right) \\
&= g(0,1 \mid 1)F(u,y \mid 1) - \frac{1 - F_B(y+2)}{p_{00} - \pi f_B(1)}(\pi 1_{\{u=y\}} + p_{01} 1_{\{u \le y\}})
\end{aligned}$$

which also can be rewritten as

$$\begin{aligned}
F(u+1,y \mid 1) = &\frac{\alpha}{p_{00} - \pi f_B(1)} F(u,y \mid 1) \\
&- \frac{\alpha^{y+2}}{p_{00} - \pi f_B(1)}(\pi 1_{\{u=y\}} + p_{01} 1_{\{u \le y\}}) .
\end{aligned} \tag{64}$$

Successive applications of (64) yields (63).
Moreover, for the geometric distribution, it follows, from (53), (54) and (24), that

$$F(k,y \mid 0) = \begin{cases} \dfrac{q}{1-q} \cdot \dfrac{p_{00} - \pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} \\ \quad \cdot \left(F(k,y \mid 1) - \dfrac{\pi}{p_{00} - \pi f_B(1)} \alpha^{y+2} \right), & k \le y \\ \\ \dfrac{q}{1-q} \cdot \dfrac{p_{00} - \pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} F(k,y \mid 1), & k > y \end{cases} . \tag{65}$$

Then, we obtain (62) by combining (63) and (65). □
Having separately examined the (defective) distributions of the surplus at ruin and the surplus before ruin, we now consider their joint (defective) distribution. Let us define by

$$F(u,x,y \mid i) = \Pr(T < \infty, U_{T-1} \le x, U_T \ge -y \mid I_0 = i, U_0 = u) \tag{66}$$

the probability that ruin occurs, that the surplus one period before ruin is lower or equal to x and that the surplus at ruin is not greater than y given an initial occurrence state $I_0 = i$.
The joint (defective) distribution of the surplus one period before ruin and the severity of ruin can be computed recursively as stated in the following proposition.

***Proposition 5** In the compound Markov binomial model, the joint probability $F(u,x,y \mid i)$ can be computed recursively with*

- *for $u=0$, $x \in \mathbb{N}$ and $y \in \mathbb{N}^+$,*

$$F(0,x,y \mid 0) = \frac{p_{01}}{p_{10}} \sum_{j=0}^{x} (F_B(j+y+1) - F_B(j+1));$$

$$F(0,x,y \mid 1) = \frac{p_{10} F(0,x,y \mid 0) + \pi \sum_{j=2}^{y+1} f_B(j)}{p_{00} - \pi f_B(1)};$$

- *for $u \in \mathbb{N}^+$, $x \in \mathbb{N}$ and $y \in \mathbb{N}^+$,*

$$F(u,x,y \mid 0) = \frac{F(u-1,x,y \mid 0)}{p_{00}} - p_{01} \frac{\sum_{k=1}^{u} F(u-k,x,y \mid 1) f_B(k) + 1_{\{u \le x+1\}} \sum_{k=u+1}^{u+y} f_B(k)}{p_{00}};$$

$$F(u,x,y \mid 1) = p_{10} \frac{F(u,x,y \mid 0)}{p_{00} - \pi f_B(1)} + \pi \frac{\sum_{k=2}^{u+1} F(u+1-k,x,y \mid 1) f_B(k) + 1_{\{u \le x\}} \sum_{k=u+2}^{u+y+1} f_B(k)}{p_{00} - \pi f_B(1)}.$$

Proof: Similar to the proofs of Propositions 1 and 3. □

One can use the results of the previous proposition in order to obtain $G(u,y \mid i)$ and $H(u,x \mid i)$ for $u \in \mathbb{N}^+$, $x \in \mathbb{N}$, $y \in \mathbb{N}^+$ and $i \in \{0,1\}$ since $G(u,y \mid i) = \lim_{x \to \infty} F(u,x,y \mid i)$ and $F(u,x \mid i) = \lim_{y \to \infty} F(u,x,y \mid i)$. For example, we have

$$\begin{aligned}
G(0,y \mid 0) &= \lim_{x \to \infty} F(0,x,y \mid 0) \\
&= \frac{p_{01}}{p_{10}} \sum_{j=0}^{\infty} (F_B(j+y+1)) - F_B(j+1)) \\
&= \frac{q}{1-q} \left(\sum_{j=0}^{\infty} (1 - F_B(j+1)) - \sum_{j=y}^{\infty} (1 - F_B(j+1)) \right) \\
&= \frac{q}{1-q} (E[B \wedge (y+1)] - 1),
\end{aligned}$$

and

$$
\begin{aligned}
F(0,x \mid 0) &= \lim_{y\to\infty} F(0,x,y \mid 0) \\
&= \frac{p_{01}}{p_{10}} \lim_{y\to\infty} \sum_{j=0}^{x} (F_B(j+y+1) - F_B(j+1)) \\
&= \frac{q}{1-q} \sum_{j=0}^{x} (1 - F_B(j+1)) \\
&= \frac{q}{1-q} (E[B \wedge (x+2)] - 1) .
\end{aligned}
$$

5 Distribution of the claim causing ruin

Another quantity of interest to improve our knowledge of the surplus process is the distribution of the claim amount that caused ruin. Clearly, in discrete-time risk models, the claim causing ruin is equal to the sum of the surplus a period prior ruin, the premium income for the period in which the ruin occurs and the surplus at ruin. The distribution of the claim amount causing ruin has been first studied by Dufresne and Gerber (1988) and Dickson (1993) in the context of the classical risk model.

Therefore, let us denote respectively by $H(u, y \mid i)$ and $H(u, y)$ the conditional and unconditional probability that ruin occurs and that the claim that caused ruin is lower or equal to y which are defined as

$$
H(u, y \mid i) = \Pr(T < \infty, U_{T-1} + 1 - U_T \leq y \mid I_0 = i) ,
$$

and

$$
H(u, y) = \Pr(T < \infty, U_{T-1} + 1 - U_T \leq y) ,
$$

for $u \in \mathbb{N}$, $y \in \{2, 3, \dots\}$ and $i \in \{0, 1\}$. We have

$$
H(u, y) = (1-q) H(u, y \mid 0) + q H(u, y \mid 1) .
$$

In the following proposition, we present a recursive algorithm to compute the conditional probabilities $H(u, y \mid i)$.

Proposition 6 *In the compound Markov binomial model, the conditional (defective) distribution of the claim causing ruin is given by*

- *for* $u = 0$ *and* $y \in \{2, 3, \ldots\}$,

$$H(0, y \mid 0) = \frac{q}{1-q} \sum_{j=0}^{y-2} \sum_{k=j+2}^{y} f_B(k) \, ; \tag{67}$$

$$H(0, y \mid 1) = \frac{p_{10} H(0, y \mid 0) + \pi \sum_{j=2}^{y} f_B(j)}{p_{00} - \pi f_B(1)} \, ; \tag{68}$$

- *for* $u \in \mathbb{N}^+$ *and* $y \in \{2, 3, \ldots\}$,

$$H(u, y \mid 0) = \frac{H(u-1, y \mid 0)}{p_{00}} - p_{01} \frac{\sum_{k=1}^{u} H(u-k, y \mid 1) f_B(k) + 1_{\{u \leq y-1\}} \sum_{k=u+1}^{y} f_B(k)}{p_{00}} \, ; \tag{69}$$

$$H(u, y \mid 1) = \frac{p_{10} H(u, y \mid 0)}{p_{00} - \pi f_B(1)} + \pi \frac{\sum_{j=2}^{u+1} H(u+1-j, y \mid 1) f_B(j) + 1_{\{u \leq y-2\}} \sum_{j=u+2}^{y} f_B(j)}{p_{00} - \pi f_B(1)} \, . \tag{70}$$

Proof: The proof is similar to the ones of the previous propositions which explains why we only outline the major steps.
By first conditioning on the claim occurrence and claim amount (if necessary) r.v.'s in the first period (i.e. I_1 and B_1) and given the stationarity of the surplus process, one finds

$$H(j, y \mid i) = p_{i0} H(j+1, y \mid 0) + p_{i1} \left(\sum_{k=1}^{j+1} H(j+1-k, y \mid 1) f_B(k) + 1_{\{j+2 \leq u\}} \sum_{k=j+2}^{y} f_B(k) \right) . \tag{71}$$

(69) follows easily from (71) at $i = 0$ and (70) is the direct consequence of combining (71) at $i = 0$ and $i = 1$. Moreover, (68) is obtained with the

combination of (71) at $i=1$ and $j=0$ with (71) at $i=0$ and $j=0$. It remains to prove the starting point of the recursive formula given in (67).

First, summing (71) for $j=0,1,\dots,u-1$ and given $p_{i0}=1-p_{i1}$ for $i\in\{0,1\}$, one obtains

$$H(0,y\mid 0)-H(u,y\mid 0)$$
$$=\frac{q}{1-q}\left(\sum_{j=0}^{u-1}1_{\{j+2\le y\}}\sum_{k=j+2}^{y}f_B(k)-\sum_{j=0}^{u-1}H(j,y\mid 1)(1-F_B(u-j))\right). \qquad (72)$$

Then, taking the limit $u\to\infty$, applying the dominated convergence theorem (as in Propositions 1 and 3) and since $\lim_{u\to\infty}H(u,y\mid 0)=0$ for $y\in\{2,3,\dots\}$ due to (4), we obtain the desired result. □

The algorithms proposed in Proposition 6 for the distribution of the claim causing ruin are however unstable. A stable algorithm is provided in the next proposition.

Proposition 7 *In the compound Markov binomial model, a stable algorithm to compute the conditional (defective) distribution of the claim causing ruin is*

$$H(u,y\mid 0)=\begin{cases}\displaystyle\sum_{j=1}^{u}g(0,j\mid 0)H(u-j,y\mid 1)\\ \displaystyle\quad+\sum_{j=u+1}^{y-1}\left(g(0,j\mid 0)-g(0,y\mid 0)\right), & u\le y-2\\ \displaystyle\sum_{j=1}^{u}g(0,j\mid 0)H(u-j,y\mid 1), & u>y-2\end{cases} \qquad (73)$$

and

$$H(u,y\mid 1)=\begin{cases}\displaystyle\sum_{j=1}^{u}g(0,j\mid 1)H(u-j,y\mid 1)\\ \displaystyle\quad+\sum_{j=u+1}^{y-1}\left(g(0,j\mid 1)-\frac{p_{10}g(0,y\mid 0)}{p_{00}-\pi f_B(1)}\right), & u\le y-2\\ \displaystyle\sum_{j=1}^{u}g(0,j\mid 1)H(u-j,y\mid 1), & u>y-2\end{cases} \qquad (74)$$

Proof: We begin by proving (73). First, by rearranging (72) using (67) and (24), one finds

$$
\begin{aligned}
H(u, y \mid 0) &= \frac{q}{1-q} \sum_{j=0}^{u-1} H(j, y \mid 1)(1 - F_B(u-j)) \\
&\quad + \frac{q}{1-q} \left(\sum_{j=0}^{y-2} \sum_{k=j+2}^{y} f_B(k) - \sum_{j=0}^{u-1} 1_{\{j+2 \le y\}} \sum_{k=j+2}^{y} f_B(k) \right) \\
&= \frac{q}{1-q} \sum_{j=1}^{u} H(u-j, y \mid 1)(1 - F_B(j)) \\
&\quad + \frac{q}{1-q} 1_{\{u \le y-2\}} \sum_{j=u}^{y-2} \sum_{k=j+2}^{y} f_B(k) \\
&= \sum_{j=1}^{u} g(0, j \mid 0) H(u-j, y \mid 1) \\
&\quad + 1_{\{u \le y-2\}} \sum_{j=u+1}^{y-1} \left(g(0, j \mid 0) - g(0, y \mid 0) \right). \qquad (75)
\end{aligned}
$$

By considering both cases $u \le y-2$ and $u > y-2$, the result easily follows. To prove (74), we combine (75) and (70)

$$
\begin{aligned}
H(u, y \mid 1) &= \sum_{j=1}^{u} \frac{p_{10} g(0, j \mid 0) + \pi f_B(j+1)}{p_{00} - \pi f_B(1)} H(u-j, y \mid 1) \\
&\quad + 1_{\{u \le y-2\}} \sum_{j=u+1}^{y-1} \frac{p_{10} g(0, j \mid 0) + \pi f_B(j+1)}{p_{00} - \pi f_B(1)} \\
&\quad - \frac{p_{10}}{p_{00} - \pi f_B(1)} \sum_{j=u+1}^{y-1} g(0, y \mid 0) \\
&= \sum_{j=1}^{u} g(0, j \mid 1) H(u-j, y \mid 1) + 1_{\{u \le y-2\}} \sum_{j=u+1}^{y-1} g(0, j \mid 1) \\
&\quad - 1_{\{u \le y-2\}} \frac{p_{10}}{p_{00} - \pi f_B(1)} \sum_{j=u+1}^{y-1} g(0, y \mid 0). \qquad (76)
\end{aligned}
$$

□

Let us define by $h(u, y \mid i) = \Pr(T < \infty, U_{T-1} + 1 - U_T = y \mid I_0 = i)$ the probability that the claim causing ruin is equal to y for $y \in \{2, 3, \ldots\}$ given that $u \in \mathbb{N}$ and $I_0 = i$. It follows that

$$h(u, y \mid i) = \begin{cases} H(u, y \mid i) - H(u, y-1 \mid i), & y \in \{3, 4, \ldots\} \\ H(u, y \mid i), & y = 2 \end{cases}.$$

In particular, assuming an initial surplus u equal to 0, one deduces, from Proposition 6, that

$$\begin{aligned} h(0, y \mid 0) &= \frac{q}{1-q} \sum_{j=0}^{y-2} \sum_{k=j+2}^{y} f_B(k) - \frac{q}{1-q} \sum_{j=0}^{y-3} \sum_{k=j+2}^{y-1} f_B(k) \\ &= \frac{q}{1-q} (y-1) f_B(y) \end{aligned}$$

and

$$h(0, y \mid 1) = \frac{p_{01}(y-1) f_B(y) + \pi f_B(y)}{p_{00} - \pi f_B(1)},$$

for $y \in \{2, 3, \ldots\}$.
Cases for which an explicit expression can be found for the distribution of the claim causing ruin are presented in the following examples.

Example 5 *(continuation of Example 1) The claim amount r.v. B takes values in $\{1, 2\}$ with $f_B(2) > 0$. For this special case, a closed-form expression for the (defective) distribution of the claim causing ruin is given by*

$$H(u, y \mid i) = \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} \right)^u H(0, y \mid i), \tag{77}$$

for $u \in \mathbb{N}^+$, $y \in \{2, 3, \ldots\}$ and $i \in \{0, 1\}$.

Proof: From (29), (74) becomes

$$\begin{aligned} H(u, y \mid 1) &= g(0, 1 \mid 1) H(u-1, y \mid 1) \\ &= \frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} H(u-1, y \mid 1), \end{aligned} \tag{78}$$

for $u \in \mathbb{N}^+$ and $y \in \{2, 3, \ldots\}$. Successive applications of (78) yields (77) for $i = 1$.

Similarly, by combining (32) and (73), one has

$$H(u,y \mid 0) = \frac{q f_B(2)}{1-q} H(u-1,y \mid 1) \tag{79}$$

for $u \in \mathbb{N}^+$ and $y \in \{2,3,\ldots\}$. From (77) at $i=1$, (79) becomes

$$H(u,y \mid 0) = \frac{q f_B(2)}{1-q} \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} \right)^{u-1} H(0,y \mid 1)\,. \tag{80}$$

In the case where $B \in \{1,2\}$, (67) and (68) are respectively given by $H(0,y \mid 0) = \frac{q}{1-q} f_B(2)$ and $H(0,y \mid 1) = \frac{p_{11}}{p_{00} - \pi f_B(1)} f_B(2)$ which permits to rewrite (80) as

$$H(u,y \mid 0) = \left(\frac{p_{11} f_B(2)}{p_{00} - \pi f_B(1)} \right)^{u} H(0,y \mid 0)\,.$$

This completes the proof of (77). □

Example 6 *(continuation of Example 2) When the claim amount r.v. B has a geometric distribution, an explicit expression for the conditional distribution of the claim causing ruin can be found. Since this expression is cumbersome, we present instead the two following relations of interest*

$$\begin{aligned} H(u+1,y \mid 1) = {} & \frac{\alpha}{p_{00} - \pi f_B(1)} H(u,y \mid 1) - 1_{\{u \le y-2\}} g(0,y \mid 1) \\ & + 1_{\{u \le y-2\}} \frac{p_{10}}{p_{00} - \pi f_B(1)} \alpha \sum_{j=u+1}^{y-1} g(0,y \mid 0) \\ & - 1_{\{u \le y-3\}} \frac{p_{10}}{p_{00} - \pi f_B(1)} \sum_{j=u+2}^{y-1} g(0,y \mid 0)\,, \end{aligned} \tag{81}$$

and

$$H(k,y \mid 0) = \begin{cases} \dfrac{q}{1-q} \cdot \dfrac{p_{00} - \pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} H(k,y \mid 1)\,, & y < k+2 \\[2ex] \dfrac{q}{1-q} \cdot \dfrac{p_{00} - \pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} H(k,y \mid 1) & \\ \quad - \dfrac{\pi(1-\alpha)}{p_{01} + \pi(1-\alpha)} \displaystyle\sum_{j=u+1}^{y-1} g(0,y \mid 0)\,, & y \ge k+2 \end{cases} \tag{82}$$

Successive applications of (81) *yield a closed-form expression for* $H(u, y \mid 1)$ *which implies, from* (82), *that* $H(u, y \mid 0)$ *has also an explicit expression.* □

6 Numerical example

To illustrate the theoretical results obtained in the previous sections, consider the following example. Assume that the claim amount r.v. B is geometrically distributed with p.m.f. $f_B(i) = (1-\alpha)\alpha^{i-1}$ for $i \in \mathbb{N}^+$ and mean 10. Moreover, suppose that $\Pr(I_k = 1) = q = 0.08$ and that the relative security margin η is equal to 25 %. We consider three cases for π: $\pi = 0$ (independence), $\pi = 0.4$ and $\pi = 0.8$ from which one deduces, in each case, the transition probability matrix $\underline{P}$ of (3). Exact values for the distributions of the severity of ruin, the surplus one period prior ruin and the claim causing ruin are given in the following tables (see pp. 110–112).

The results contained in Tables 1 to 4 and in Figures 1 and 2 allow the following observations and comments:

- Based on the results presented in Tables 1 to 4, one could be inclined to think that, for a given initial surplus u and a given y, the values of $G(u, y)$, $F(u, y)$ and $H(u, y)$ increase with π. Other choices of distributions for the claim amount B doesn't necessarily lead to that conclusion.

- In Tables 1 to 4, the values of $G(u, y)$, $F(u, y)$ and $H(u, y)$ all increase with u for a given value of π and a given y. This observation doesn't necessarily hold for other choices of distributions for the claim amount r.v. B. This is confirmed by Figures 1 and 2.

- For a given point y of any of the distributions, the impact of π seems more influent for high initial surplus levels. It can be explained by the fact that the ruin probabilities seem more sensitive to the dependence parameter π for high initial surplus values since, for low initial surplus values, ruin events occur more independently of π due to its proximity to the ruin barrier. The impact of π is thus minimized when $u = 0$ for the three distributions considered.

- The values of $G(u, y)$, $F(u, y)$ and $H(u, y)$ converge all to the same numerical value as $y \to \infty$ i.e.

$$\lim_{y \to \infty} G(u, y) = \lim_{y \to \infty} F(u, y) = \lim_{y \to \infty} H(u, y) = \psi(u) .$$

- Figures 1 and 2 confirm our intuition that the distribution of the claim causing ruin given that ruin occurred should be more dangerous on stochastic order than the individual claim amount distribution. On average, the amount of the claim causing ruin given that ruin occurred is larger than a random individual claim amount.

Table 1. The (defective) distribution of the severity of ruin from an initial surplus of 0 for $\pi = 0$, 0.4 and 0.8.

	$G(u, y \mid 0)$		$G(u, y \mid 1)$		$G(u, y)$		
y/π	0.4	0.8	0.4	0.8	0	0.4	0.8
1	0.07826	0.07826	0.08684	0.09558	0.07826	0.07895	0.07965
2	0.14870	0.14870	0.16500	0.18159	0.14870	0.15000	0.15133
3	0.21209	0.21209	0.23534	0.25901	0.21209	0.21395	0.21584
4	0.26914	0.26914	0.29865	0.32868	0.26914	0.27150	0.27390
5	0.32049	0.32049	0.35563	0.39139	0.32049	0.32330	0.32616
10	0.50973	0.50973	0.56562	0.62250	0.50973	0.51420	0.51875
15	0.62148	0.62148	0.68962	0.75897	0.62148	0.62693	0.63248
20	0.68746	0.68746	0.76284	0.83956	0.68746	0.69349	0.69963
25	0.72643	0.72643	0.80608	0.88714	0.72643	0.73280	0.73928
30	0.74943	0.74943	0.83161	0.91524	0.74943	0.75601	0.76270
40	0.77104	0.77104	0.85559	0.94163	0.77104	0.77780	0.78469
50	0.77858	0.77858	0.86395	0.95083	0.77858	0.78540	0.79236
60	0.78120	0.78120	0.86686	0.95403	0.78120	0.78805	0.79503
80	0.78244	0.78244	0.86823	0.95554	0.78244	0.78930	0.79629
100	0.78259	0.78259	0.86840	0.95573	0.78259	0.78945	0.79644
200	0.78261	0.78261	0.86842	0.95575	0.78261	0.78947	0.79646
500	0.78261	0.78261	0.86842	0.95575	0.78261	0.78947	0.79646

Table 2. The (defective) distribution of the severity of ruin from an initial surplus of 20 for $\pi = 0$, 0.4 and 0.8.

	$G(u, y \mid 0)$		$G(u, y \mid 1)$		$G(u, y)$		
y/π	0.4	0.8	0.4	0.8	0	0.4	0.8
1	0.06005	0.07162	0.06663	0.08746	0.05042	0.06057	0.07289
2	0.11409	0.13608	0.12660	0.16618	0.09581	0.11509	0.13848
3	0.16273	0.19409	0.18057	0.23703	0.13665	0.16416	0.19752
4	0.20650	0.24630	0.22915	0.30079	0.17341	0.20832	0.25066
5	0.24590	0.29329	0.27287	0.35817	0.20649	0.24806	0.29848
10	0.39111	0.46647	0.43399	0.56967	0.32842	0.39454	0.47472
15	0.47685	0.56873	0.52913	0.69456	0.40042	0.48103	0.57880
20	0.52748	0.62912	0.58531	0.76830	0.44294	0.53210	0.64025
25	0.55737	0.66477	0.61849	0.81185	0.46804	0.56226	0.67654
30	0.57502	0.68583	0.63808	0.83756	0.48287	0.58007	0.69797
40	0.59160	0.70560	0.65647	0.86171	0.49679	0.59679	0.71809
50	0.59738	0.71250	0.66289	0.87013	0.50164	0.60262	0.72511
60	0.59940	0.71490	0.66512	0.87306	0.50334	0.60466	0.72755
80	0.60035	0.71603	0.66618	0.87444	0.50413	0.60561	0.72870
100	0.60046	0.71617	0.66630	0.87461	0.50423	0.60573	0.72884
200	0.60048	0.71619	0.66632	0.87464	0.50424	0.60575	0.72886
500	0.60048	0.71619	0.66632	0.87464	0.50424	0.60575	0.72886

Table 3. The (defective) distribution of the surplus one period prior ruin for $\pi = 0$, 0.4 and 0.8.

	$F(u, y)$					
	$u = 0$			$u = 20$		
y/π	0	0.4	0.8	0	0.4	0.8
1	0.14870	0.17558	0.20294	0.01138	0.05450	0.11429
2	0.21209	0.23697	0.26229	0.02140	0.08182	0.16479
3	0.26914	0.29222	0.31571	0.03356	0.10885	0.21129
4	0.32049	0.34194	0.36378	0.04739	0.13540	0.25409
5	0.36670	0.38670	0.40705	0.06250	0.16134	0.29347
10	0.53702	0.55164	0.56652	0.14693	0.27851	0.44754
15	0.63759	0.64903	0.66068	0.22967	0.37162	0.54812
20	0.69698	0.70655	0.71628	0.30896	0.45045	0.62206
25	0.73204	0.74051	0.74912	0.38893	0.51404	0.66580
30	0.75275	0.76056	0.76850	0.43615	0.55160	0.69162
40	0.77220	0.77939	0.78671	0.48050	0.58687	0.71588
50	0.77898	0.78596	0.79306	0.49596	0.59916	0.72434
60	0.78134	0.78825	0.79528	0.50135	0.60345	0.72728
80	0.78245	0.78932	0.79632	0.50389	0.60547	0.72867
100	0.78259	0.78946	0.79644	0.50420	0.60571	0.72884
200	0.78261	0.78947	0.79646	0.50424	0.60575	0.72886
500	0.78261	0.78947	0.79646	0.50424	0.60575	0.72886

Table 4. The (defective) distribution of the claim causing ruin for $\pi = 0$, 0.4 and 0.8.

	$H(u, y)$					
	$u = 0$			$u = 20$		
y/π	0	0.4	0.8	0	0.4	0.8
2	0.00783	0.01074	0.01370	0.00040	0.00271	0.00595
3	0.02191	0.02722	0.03262	0.00150	0.00789	0.01678
4	0.04093	0.04820	0.05559	0.00349	0.01529	0.03158
5	0.06375	0.07260	0.08160	0.00650	0.02464	0.04955
10	0.20653	0.21935	0.23240	0.03767	0.09273	0.16606
15	0.35292	0.36613	0.37958	0.09082	0.17855	0.29219
20	0.47602	0.48830	0.50080	0.15498	0.26501	0.40405
25	0.57036	0.58141	0.59265	0.22683	0.34769	0.49762
30	0.63885	0.64876	0.65886	0.29840	0.41994	0.56933
40	0.71963	0.72796	0.73645	0.40316	0.51765	0.65721
50	0.75617	0.76369	0.77134	0.45878	0.56690	0.69829
60	0.77183	0.77897	0.78624	0.48482	0.58937	0.71625
80	0.78092	0.78783	0.79486	0.50101	0.60307	0.72686
100	0.78236	0.78923	0.79622	0.50374	0.60534	0.72856
200	0.78261	0.78947	0.79646	0.50424	0.60575	0.72886
500	0.78261	0.78947	0.79646	0.50424	0.60575	0.72886

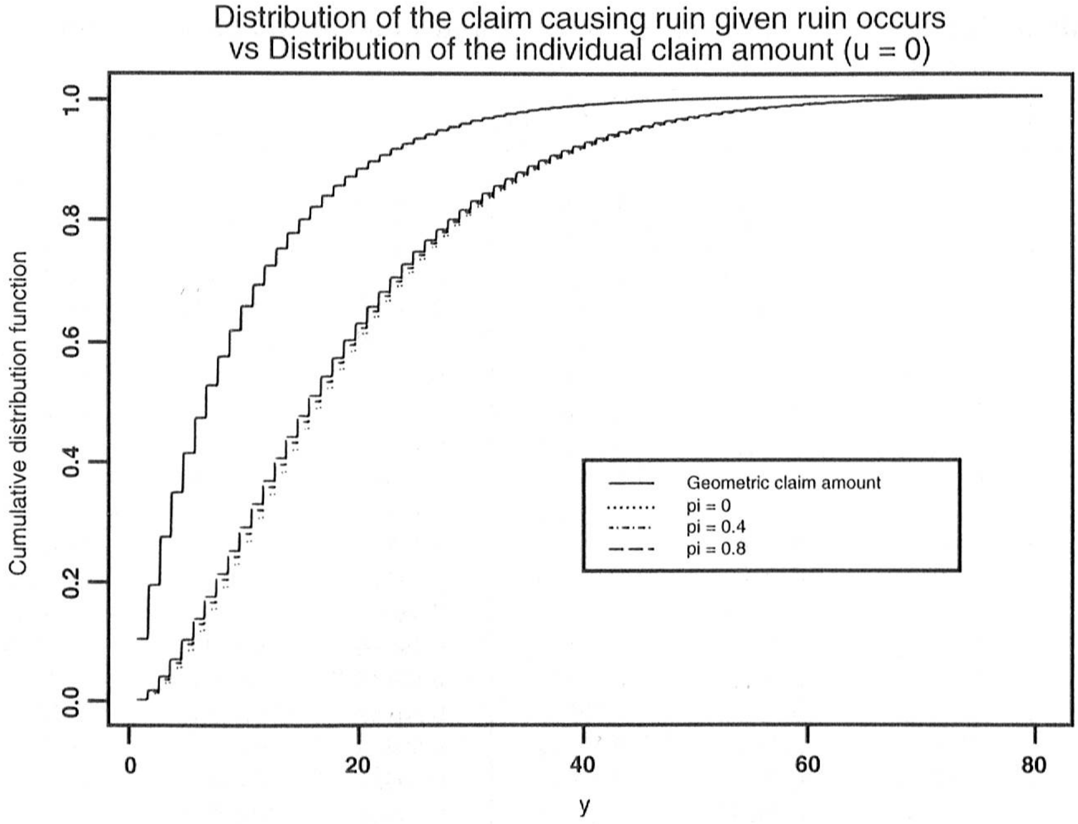

Figure 1

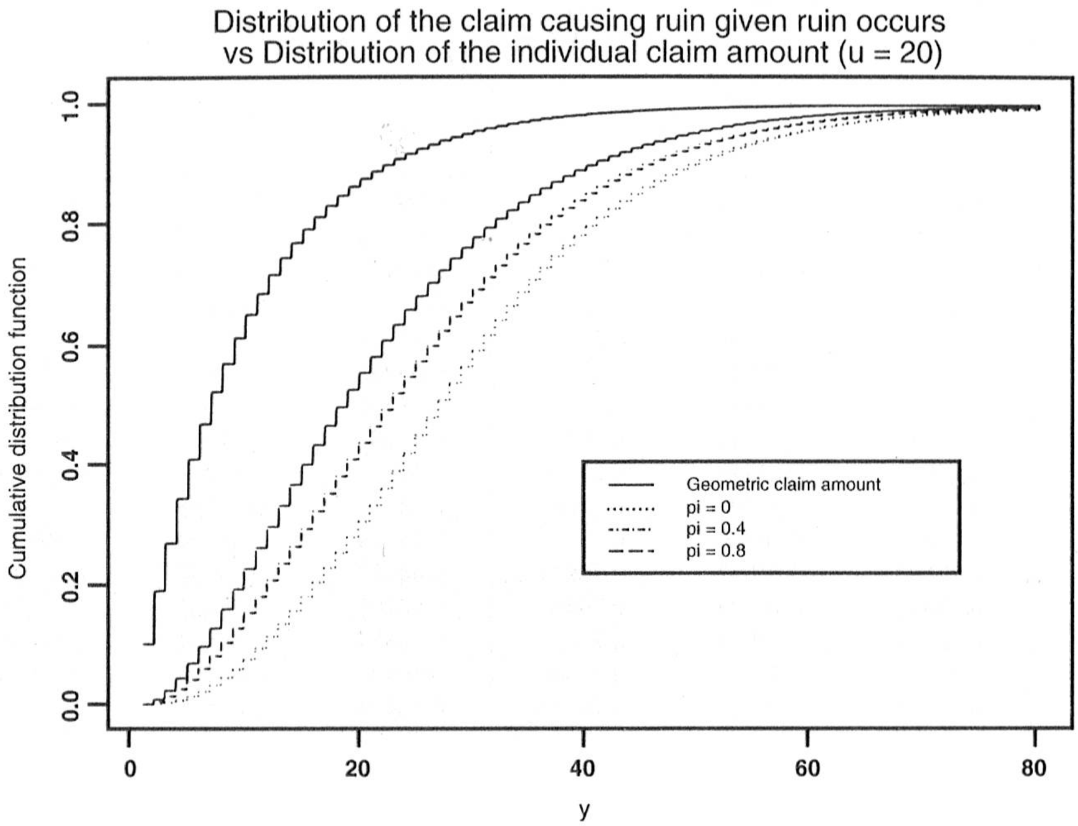

Figure 2

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Abstract

Gerber (1988a,b) has proposed a compound binomial model, as an approximation to the classical risk model, to describe the surplus process of an insurance company. Within the compound binomial model, the claims occur according to a binomial process with independent increments. Cossette et al. (2003) present a compound Markov binomial model which is an extension of Gerber's model. The compound Markov binomial model is based on a Markov binomial process which introduces dependency between claim occurrences over time. In this paper, we study, in details, some properties of the surplus process within the compound Markov binomial model. Recursive formulas for the computation of the distribution of the severity of ruin and the surplus one period prior to ruin are provided. Finally, we examine the computation of the joint distribution of the surplus prior and after the ruin and the distribution of the claim causing ruin.

Résumé

Gerber (1988a,b) a proposé le modèle binomial composé pour décrire le processus de surplus d'une compagnie d'assurance. Ce modèle en temps discret peut être utilisé pour approximer le modèle classique de risque qui est basé sur le processus Poisson composé. Dans le cadre du modèle binomial composé, les sinistres surviennent selon un processus binomial avec incréments indépendants. Cossette et al. (2003) a proposé le modèle Markov binomial composé basé sur un processus Markov binomial introduisant une relation de dépendance entre la survenance des sinistres. Dans ce papier, on étudie, en détails, certaines propriétés du processus de surplus dans le modèle Markov binomial composé. Des algorithmes récursifs sont présentés pour calculer la distribution de la sévérité de la ruine et celle du surplus une période avant la ruine. Pour conclure, on étudie aussi l'évaluation de la distribution conjointe de la sévérité de la ruine et du surplus une période avant la ruine ainsi que la distribution du montant du sinistre qui cause la ruine.

Zusammenfassung

Gerber (1988a,b) hat ein zusammengesetztes Binomial-Modell als Approximation zum klassischen Risikomodell vorgeschlagen, um den Ergebnisprozess einer Versicherungsgesellschaft zu beschreiben. Im zusammengesetzten Binomial-Modell treten die einzelnen Schaden gemäss eines Binomialprozesses mit unabhängigen Zuwächsen auf. Cossette et al. (2003a) haben ein zusammengesetztes Markov Binomial-Modell vorgeschlagen, das eine Erweiterung von Gerbers Modell darstellt. Das zusammengesetzte Markov Binomial-Modell stützt sich auf einen Markov Binomialprozess, der eine Abhängigkeit zwischen den Schadeneintrittszeiten einführt. In diesem Artikel studieren wir detailliert einige Eigenschaften des Ergebnisprozesses im zusammengesetzten Markov Binomial-Modell. Es werden rekursive Formeln zur Berechnung der Verteilung der Höhe des Ruins, sowie des Ergebnisses eine Periode vor dem Ruin angegeben. Schliesslich wird noch die gemeinsame Verteilung der Ergebnisse vor und nach einem Ruin sowie die Verteilung des Schadens, der zum Ruin führte, untersucht.