Optimization of a chain of excess-of-loss reinsurance layers with aggregate stop-loss limits

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WERNER HÜRLIMANN, Zurich

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1 Introduction

Excess-of-loss reinsurance is one of the most often encountered non-proportional reinsurance contract in practice. In most situations, these treaties are subject to various constraints like reinstatements, annual aggregate deductibles, etc.

In several common non-life insurance risk categories like Motor Third Party Liability (MTPL), the original risk is subdivided into a retained part and several excess-of-loss reinsurance layers. Under an adverse development of the number of claims in a given reinsurance layer, the liability payment can be rather large and exceed the economic capital foreseen to cover such layers. The mentioned financial problem could be solved using paid reinstatements. However, at least two difficulties remain. The pricing of excess-of-loss reinsurance with reinstatements is not a trivial task (e.g. Hürlimann(2005)), and the optimal choice of the required number of reinstatements must be discussed. Instead of reinstatements, a better economical way might be to limit the aggregate claims of a reinsurance layer and transfer the corresponding stop-loss risk to the reinsurance market or any other third party.

In the present study, we consider a chain of excess-of-loss layers with given deductibles, each with an aggregate stop-loss limit. Given the structure of the excess-of-loss deductibles of these layers, our aim is the algorithmic numerical evaluation of uniquely defined stop-loss limits, which are optimal in the following sense. We look at the expected cost of the total retained risk of a fixed number of layers and minimize this quantity. Simultaneously, we look at the maximum cost of the total retained risk of the same number of layers and limit this quantity to the value-at-risk of the stop-loss risk of the highest layer. Pricing the transferred stop-loss risk according to some specific quantile premium calculation principle, uniquely defined optimal stop-loss limits are obtained. It turns out that the optimal stop-loss limits are equal to the unexpected losses of the stop-loss risks of the layers under the value-at-risk measure. The considered family of quantile premium principles is in so far flexible as it allows different applications. For example, it may be used to design optimal risk structures for corporate clients or it may be used in solvency testing for regulatory purposes (Swiss Solvency Test for example).

To determine in a fast way numerical stable values of the optimal stop-loss limits in practice, we use analytical approximations of the distributions of the layer risks. We notice that solving the optimal stop-loss limit equations through the application of a Monte Carlo simulation method requires usually very large sample sizes for excess-of-loss layers, whose quantile stop-loss premiums are based on high confidence levels, or quite advanced resampling techniques, which go beyond the scope of practical needs.

The paper is organized as follows. Section 2 introduces the considered chain of excess-of-loss reinsurance together with appropriately defined notations. Section 3 presents the optimal stop-loss limit equations. The approximate optimal stop-loss limits are obtained in Section 4 under approximations of the aggregate claims distributions through gamma distributions. The usual approach from the standard literature on risk theory is applied to the lowest retained layer risk. For the remaining excess-of-loss layers, we propose to approximate the distribution of the claim size through a combined four parameter exponential Pareto distribution. This choice is analytically tractable and fits real data quite well, at least for the data sets used in our practical work. Moreover, it is in line with the long-year tradition of using Pareto distributions in practice and it is consistent with the theoretical results from Extreme Value Theory. Finally, Section 5 presents a numerical example, which is based on a real-life portfolio of Property and Liability Non-Life insurance risks.

2 Excess-of-loss reinsurance structure with aggregate stop-loss limits

In the framework of the classical *collective model* of risk theory, the *aggregate claims* of a portfolio of insurance risks are described by the random variable

$$X = \sum_{i=1}^{N} Y_i \,, \tag{2.1}$$

where the *claim sizes* Y_i are independent and identically distributed and independent from the random *claim number* N. It is assumed that the random variables Y_i are non-negative.

An *excess-of-loss* or *XL-reinsurance* treaty with *deductible* d on a portfolio of risks covers for each claim Y_i the *excess claim size* $(Y_i - d)_+$, $i = 1, \dots, N$. In this setting, the *aggregate claims of the XL-reinsurance* are described by the random variable denoted by

$$X(d) = \sum_{i=1}^{N} (Y_i - d)_+ \,. \tag{2.2}$$

In the present paper we assume that the portfolio of insurance risks is structured in a chain of $m + 1$ XL-reinsurance layers. The deductibles defining the chain are denoted by $d_0 = 0 < d_1 < \cdots < d_k < d_{k+1} < \cdots < d_m < d_{m+1} = \infty$. Since Y_i is non-negative, one notes that $X(d_0) = X$. The risk of the k-th layer, $k = 0, \ldots, m$, is defined and denoted by

$$X_{k,k+1} = X(d_k) - X(d_{k+1}) = \sum_{i=1}^{N} \left\{ (Y_i - d_k)_+ - (Y_i - d_{k+1})_+ \right\}. \tag{2.3}$$

The risk of the 0-th layer represents the retained risk, which is not subject to a transfer of risk, and is given by

$$X_{0,1} = X - X(d_1) = \sum_{i=1}^{N} \left\{ Y_i - (Y_i - d_1)_+ \right\} = \sum_{i=1}^{N} \min(Y_i, d_1) \,. \tag{2.4}$$

The risk of the m-th layer represents an XL-reinsurance treaty with unlimited capacity and is given by

$$X_{m,m+1} = X(d_m) = \sum_{i=1}^{N} (Y_i - d_m)_+ \,. \tag{2.5}$$

Generalizing the expression (2.4) the total risk up to the k-th layer is defined and denoted by

$$\begin{aligned} X_{0,k} &= X - X(d_k) = \sum_{i=1}^{N} \left\{ Y_i - (Y_i - d_k)_+ \right\} \\ &= \sum_{i=1}^{N} \min(Y_i, d_k) \,, \quad k = 0, \ldots, m+1 \,. \end{aligned} \tag{2.6}$$

In particular, the extreme cases $X_{0,0} = 0$ and $X_{0,m+1} = X$ are used for ease of notation. Since the independence assumptions are preserved under the performed transformations of the claim sizes, all the expressions (2.2) to (2.6) are again collective models of risk theory.

Table 2.1: excess-of-loss reinsurance structure with aggregate stop-loss limits

Layer	XL deductible	Total retained risk	SL limit	Transferred risk	SL premium
3	$d_4 = \infty$	$X_{0,3} + \min\{X_{3,4}, L_3\}$	L_3	$(X_{3,4} - L_3)_+$	P_3
2	d_3	$X_{0,2} + \min\{X_{2,3}, L_2\}$	L_2	$(X_{2,3} - L_2)_+$	P_2
1	d_2	$X_{0,1} + \min\{X_{1,2}, L_1\}$	L_1	$(X_{1,2} - L_1)_+$	P_1
0	d_1	$X_{0,0} + \min\{X_{0,1}, L_0\}$	L_0	$(X_{0,1} - L_0)_+$	P_0

In practice, one is interested in risk structures for which the risk of the k-th layer is limited to a fixed amount L_k, $k = 0, \ldots, m$, called *stop-loss* or *SL-limit*. The remaining risk represents a *stop-loss* or *SL-reinsurance* treaty with *limit* L_k, whose liability is transferred to a reinsurer or any other third party and is described by the random variable $(X_{k,k+1} - L_k)_+$, $k = 0, \ldots, m$. Given a fixed XL-reinsurance structure $d_0 = 0 < d_1 < \cdots < d_k < d_{k+1} < \cdots < d_m < d_{m+1} = \infty$, one is interested in finding *optimal SL-limits* with respect to some decision criterion. To analyze this optimization problem, we will consider the total retained risk of the first k layers, which is described and denoted by

$$\begin{aligned} X_k &= X(d_k, d_{k+1}, L_k) \\ &= X - X(d_k) + \min\{X(d_k) - X(d_{k+1}), L_k\} \\ &= X_{0,k} + \min\{X_{k,k+1}, L_k\}, \quad k = 0, ..., m. \end{aligned} \tag{2.7}$$

The transferred stop-loss risk of the k-th layer is taken up by the reinsurance market or a third party for a *stop-loss* or *SL-premium* calculated according to a given premium calculation principle $P[\cdot]$ and denoted by

$$P_k = P(d_k, d_{k+1}, L) = P[(X_{k,k+1} - L_k)_+], \quad k = 0, \ldots, m. \tag{2.8}$$

The considered risk structure is summarized in Table 2.1 for the situation $m = 4$ encountered quite often in practice.

3 Optimal stop-loss limits for the quantile premium principle

In order to optimize the stop-loss limits by given XL deductible structure, we consider the expected cost of the total retained risk of the first k layers, which is given by

$$\begin{aligned}
C_k &= C\left(d_k, d_{k+1}, L_k\right) \\
&= E[X_{0,k}] + E[\min\left\{X_{k,k+1}, L_k\right\}] + P_k \\
&= E\left[X_{0,k+1}\right] + P\left[\left(X_{k,k+1} - L_k\right)_+\right] \\
&\quad - E\left[\left(X_{k,k+1} - L_k\right)_+\right], \quad k = 0, \ldots, m.
\end{aligned} \tag{3.1}$$

In the following, the distribution function of a continuous random variable X is denoted by $F_X(x)$, its survival function by $\overline{F}_X(x) = 1 - F_X(x)$ and its α-quantile by $Q_X(\alpha)$, $0 < \alpha < 1$. The α-quantile of the transferred stop-loss risk of the k-th layer satisfies the equality

$$Q_{(X_{k,k+1} - L_k)_+}(\alpha) = \left(Q_{X_{k,k+1}}(\alpha) - L_k\right)_+, \quad k = 0, \ldots, m. \tag{3.2}$$

It is assumed that the SL premium of the k-th layer is set according to the following parametric family of α_k-quantile premium calculation principles

$$\begin{aligned}
P_k &= \pi_{X_{k,k+1}}\left(L_k\right) \\
&\quad + r \cdot [Q_{X_{k,k+1}}(\alpha_k) - L_k - \pi_{X_{k,k+1}}(L_k)]_+, \quad k = 0, \ldots, m,
\end{aligned} \tag{3.3}$$

where $\pi_X(x) = E\left[(X - x)_+\right] = \int_x^\infty \bar{F}_X(t)\, dt$ denotes the stop-loss transform of the random variable X and the parameter r belongs to the interval $(0, 1]$. In particular, this pricing principle takes into account the fact that the SL premium should be at least equal to the expected value of the SL risk. Let us mention two important special cases. First, setting $r = 1$, this principle is in view of (3.2) interpreted as the usual percentile premium principle to the confidence level α_k. In this situation higher confidence levels are chosen for the higher layers. In practice, higher layers are hit more infrequently and the return periods of individual losses increase in the higher layers. This means that the payback periods to fund higher layers also increase. Should the payback periods remain constant for each layer, there is a need for setting higher loadings on higher layers. In this situation, charging higher loadings is equivalent with increasing confidence levels. For example, a plausible choice in the practical situation of Table 2.1 could be $\alpha_0 = 80\%$, $\alpha_1 = 95\%$, $\alpha_2 = 99\%$, $\alpha_3 = 99.9\%$. This pricing principle has been used in practice to settle an optimal risk structure for a corporate insurance risk business. Empirically, the choice $\alpha_0 = 80\%$ yields premiums for the retained risk, which are in the range of a standard deviation premium with an approximate loading factor of 50%. Second, as in the Swiss Solvency Test (SST), an alternative model is to charge only the cost-of-capital on the value-at-risk of the stop-loss risk, where the parameter r is the cost-of-capital rate, and α_k is a chosen security level, say $\alpha_k = 99\%$. This second pricing principle is in line with the new developments in solvency testing used for regulatory purposes.

Inserting (3.3) into (3.1) the *expected cost of the total retained risk of the first* k *layers* can be rewritten as

$$\begin{aligned} C_k &= E\left[X_{0,k+1}\right] \\ &\quad + r \cdot \left(Q_{X_{k,k+1}}(\alpha_k) - L_k - \pi_{X_{k,k+1}}(L_k)\right)_+, \quad k = 0, \ldots, m. \end{aligned} \tag{3.4}$$

On the other side, it is also worthwhile to look at *the maximum cost of the total retained risk of the first* k *layers*, which is defined by

$$\begin{aligned} C_k^{\max} &= L_k + P_k \\ &= L_k + \pi_{X_{k,k+1}}(L_k) \\ &\quad + r \cdot \left(Q_{X_{k,k+1}}(\alpha_k) - L_k - \pi_{X_{k,k+1}}(L_k)\right)_+, \quad k = 0, \ldots, m. \end{aligned} \tag{3.5}$$

Minimizing the expected cost under the restriction that the maximum cost remains bounded by the value-at-risk of the stop-loss risk of the last layer yields uniquely defined aggregate stop-loss limits for a given XL reinsurance deductible structure. This is the main result of the present study.

Theorem 3.1 *Given is a portfolio of insurance risks structured in a chain of* $m+1$ *XL-reinsurance layers with deductibles* $d_0 = 0 < d_1 < \ldots < d_k < d_{k+1} < \ldots < d_m < d_{m+1} = \infty$. *It is assumed that the risk of the k-th layer is limited to a fixed amount* L_k *and that the transferred stop-loss risk* $(X_{k,k+1} - L_k)_+$ *is priced according to the* α_k*-quantile premium calculation principle* (3.3), $k = 0, \ldots, m$. *If one minimizes the expected cost of the total retained risk* (3.4) *under the restriction that the maximum cost of the total retained risk* (3.5) *is bounded by the value-at-risk of the stop-loss risk of the k-th layer, then the uniquely defined optimal stop-loss limits satisfy the equations*

$$L_k + \pi_{X_{k,k+1}}(L_k) = Q_{X_{k,k+1}}(\alpha_k), \quad k = 0, \ldots, m. \tag{3.6}$$

Proof Let us first minimize the expected cost function (3.4). By definition of the stop-loss transform of a random variable X, its first derivative is given by $\frac{d}{dx}\pi_X(x) = -\overline{F}_X(x)$. We distinguish between two cases.

Case 1: $L_k + \pi_{X_{k,k+1}}(L_k) < Q_{X_{k,k+1}}(\alpha_k)$
Since $\frac{\partial C_k}{\partial L_k} = r \cdot [-1 + \overline{F}_{X_{k,k+1}}(L_k)] < 0$, the expected cost function decreases monotonically and it exceeds always the quantity $E[X_{0,k+1}]$, $k = 0, \ldots, m$.

Case 2: $L_k + \pi_{X_{k,k+1}}(L_k) \geq Q_{X_{k,k+1}}(\alpha_k)$
In this situation the expected cost function always attains its minimum and it is constantly equal to $E[X_{0,k+1}]$, $k = 0, \ldots, m$.

Now, let us look at the maximum cost function (3.5). Again, we distinguish between two cases.

Case 1: $L_k + \pi_{X_{k,k+1}}(L_k) \leq Q_{X_{k,k+1}}(\alpha_k)$
One has $\frac{\partial C_k^{\max}}{\partial L_k} = (1 - r) \cdot [1 - \overline{F}_{X_{k,k+1}}(L_k)] \geq 0$, hence the maximum cost function increases monotonically but it satisfies always the required inequality constraint $C_k^{\max} \leq Q_{X_{k,k+1}}(\alpha_k)$, where equality is attained when $r = 1$.

Case 2: $L_k + \pi_{X_{k,k+1}}(L_k) > Q_{X_{k,k+1}}(\alpha_k)$
Since $\frac{\partial C_k^{\max}}{\partial L_k} = 1 - \overline{F}_{X_{k,k+1}}(L_k) > 0$, the maximum cost function increases monotonically and it satisfies the inequality $C_k^{\max} > Q_{X_{k,k+1}}(\alpha_k)$, $k = 0, \ldots, m$, which implies that the required inequality constraint is not satisfied.
Combining all above cases yields the resulting optimal equations (3.6). ◇

Remark 3.1 The proof shows the following fact. In the limiting case $r = 1$ one has $C_k^{\max} \geq Q_{X_{k,k+1}}(\alpha_k)$ and the equality is attained when $L_k + \pi_{X_{k,k+1}}(L_k) \leq Q_{X_{k,k+1}}(\alpha_k)$. In this situation, a more stringent optimization criterion consists to minimize simultaneously the expected and the maximum cost of the total retained risk of the first k layers. Then the equations (3.6) yield the uniquely defined optimal stop-loss limits of this alternative optimization problem, which finds application to settle an optimal risk structure for an insurance risk business (example in Section 5).

4 Optimal stop-loss limits for the XL layers

According to (2.3) the risk of the k-th layer for $k = 0, \ldots, m$, which is required to price the stop-loss risk of the k-th layer, has a distribution from a collective model of risk theory. For analytical purposes, the aggregate claims distribution of $X_{k,k+1}$ is approximated by a gamma distribution $\Gamma(a_k, b_k)$ with parameters $a_k = cv_{k,k+1}^{-2}$, $b_k = cv_{k,k+1}^{-2} \cdot \mu_{k,k+1}^{-1}$, where $\mu_{k,k+1}$ is the mean and $cv_{k,k+1}$ is the coefficient of variation of the aggregate claims $X_{k,k+1}$. It is important to note that for a Poisson claim number distribution with a sufficiently large expected number of claims, the use of the gamma approximation to the aggregate claims distribution with a gamma claim size distribution can be justified (see e.g. Hürlimann(2002)). For the retained risk layer $k = 0$ these parameters are estimated applying the usual approach from the standard literature on risk theory. For example, a compound negative binomial model with truncated claims severities is appropriate. For $k > 0$ another approach is applied.

A risk-manager or a reinsurer, which does not know the number and the size of the original claims below the deductible d_k, will not be able to analyze this risk satisfactorily. Therefore, the collective model (2.3) is not appropriate to forecast the stop-loss risk of the k-th layer. Fortunately, it is possible to construct a collective model for these layers on the basis of the collective model for the original claims such that the model contains only random variables which are observable for the reinsurer. This collective model is presented in Hess(2003) and the related literature in Hess et al.(1995), Franke and Macht(1995), Mack(1997) and Schmidt(1996/2002). In this setting, to estimate the parameters $\mu_{k,k+1}$ and $cv_{k,k+1}$ of the Gamma approximation to the aggregate claims $X_{k,k+1}$, we use a compound Poisson model with exponential Pareto severities described by the distribution

$$F(x) = \begin{cases} 1 - \exp\left(-\dfrac{x-\alpha}{\beta}\right), & \alpha \leq x \leq T, \\ 1 - \exp\left(-\dfrac{T-\alpha}{\beta}\right) \cdot \left(\dfrac{x}{T}\right)^{-\gamma}, & x \geq T. \end{cases} \tag{4.1}$$

The use of (4.1) is justified as follows. It is well-known that the two-parameter Pareto distribution is an appropriate distribution often used to fit large claims distributions in reinsurance. This has been a first choice in the practice of reinsurance for a long time (see e.g. Schmitter(1978), Schmitter and Bütikofer(1997), Doerr(1980), Schmutz and Doerr(1998)) and it is consistent with the theoretical results from Extreme Value Theory (e.g. Embrechts et al.(1997)). Once the large claims distribution has been fitted in an adequate way, one often observes a rather poor fit in the lower tail of the distribution. To remedy for this disadvantage, it appears attractive to fit the lower tail using another simple two-parameter analytical distribution, for example a translated exponential distribution, which is our choice here. To fit (4.1) to claims data, we proceed in two steps as follows. In a first step, one determines the threshold T and the Pareto index γ minimizing the chi-square statistic of the Pareto tail $\left(\frac{x}{T}\right)^{-\gamma}$, $x \geq T$, over some plausible set $[T_1, T_2]$ of threshold choices. In a second step, one determines the remaining parameters α, β such that the chi-square value and the Cramér-von Mises K-statistic are sufficiently small.

Based on the survival function $\overline{F}(x)$, the mean and coefficient of variation of the aggregate claims $X_{k,k+1}$ of the k-th layer with are given by the formulas (valid for $x > T$)

$$\mu_{k,k+1} = E[X_{k,k+1}] = \mu_N \cdot \{m(d_k) - m(d_{k+1})\}, \tag{4.2}$$

$$m(x) = E[(Y-x)_+] = \exp\left\{-\frac{T-\alpha}{\beta}\right\} \cdot \frac{T}{\gamma-1} \cdot \left(\frac{x}{T}\right)^{-(\gamma-1)}. \tag{4.3}$$

$$\begin{aligned} cv_{k,k+1} &= \frac{\sqrt{\mathrm{Var}[X_{k,k+1}]}}{\mu_{k,k+1}} \\ &= \sqrt{\mu_N \cdot \frac{Var[(Y-d_k)_+ - (Y-d_{k+1})_+]}{\mu_{k,k+1}^2} + cv_N^2}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\mathrm{Var}[(Y-d_k)_+ - (Y-d_{k+1})_+] \\ &= m2(d_k) - m2(d_{k+1}) - 2 \cdot (d_{k+1} - d_k) \cdot m(d_{k+1}) \\ &\quad - (m(d_k) - m(d_{k+1}))^2, \qquad d_{k+1} > d_k > T, \end{aligned} \tag{4.5}$$

$$\begin{aligned} m2(x) &= E[(Y-x)_+^2] \\ &= \exp\left\{-\frac{T-\alpha}{\beta}\right\} \cdot \frac{2 \cdot T^2}{(\gamma-1)(\gamma-2)} \cdot \left(\frac{x}{T}\right)^{-(\gamma-2)}. \end{aligned} \tag{4.6}$$

To evaluate the mean and ε-quantile of the stop-loss risk $(X_{k,k+1} - L_k)_+$, we use the formulas

$$\begin{aligned} \mu_k &= E[(X_{k,k+1} - L_k)_+] \\ &= \mu_{k,k+1} \cdot \{1 - \Gamma(b_k \cdot L_k, 1 + a_k)\} \\ &\quad - L_k \cdot \{1 - \Gamma(b_k \cdot L_k, a_k)\}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} Q_{(X_{k,k+1} - L_k)_+}(\varepsilon) &= (q_k(\varepsilon) - L_k)_+, \\ q_k(\varepsilon) &= Q_{X_{k,k+1}}(\varepsilon) \\ &= \Gamma^{-1}\left(\frac{1}{cv_{k,k+1}^2}; \varepsilon\right) \cdot cv_{k,k+1}^2 \cdot \mu_{k,k+1}, \end{aligned} \tag{4.8}$$

where $\Gamma(bx; a) = \frac{1}{\Gamma(a)} \cdot \int_0^{bx} t^{a-1} e^{-t} dt$ denotes the incomplete gamma function and $\Gamma^{-1}(c; \varepsilon)$ denotes the ε-quantile of the "standardized" gamma distribution $\Gamma(c, 1)$ and (3.2) has been used for determining the quantile of the stop-loss risk.

5 A numerical example

A main advantage of the proposed method is its full analytical tractability and the fast and numerical stable evaluation of all required quantities. All calculations can be done using any modern computer algebra system, for example the software package MATHCAD 13 from Mathsoft Engineering & Education, Inc. (`www.mathsoft.com`) will do.

Table 5.1 Optimal XL SL reinsurance structure (figures in units of millions)

Retained risk $(0, d_1]$						
d_1	$\mu_{0,1}$	$\sigma_{0,1}$	$cv_{0,1}$	L_0	$\pi_{0,1}(L_0)$	$Q_{0,1}$ (80%)
1.00	40.300	6.755	0.168	44.736	1.111	45.847
1.25	44.194	7.608	0.172	49.182	1.255	50.437
1.50	47.270	8.327	0.176	52.719	1.378	54.097
1.75	49.738	8.940	0.180	55.581	1.484	57.065
2.00	51.744	9.466	0.183	57.922	1.575	59.497

First layer $(d_1, d_2]$							
d_1	d_2	$\mu_{1,2}$	$\sigma_{1,2}$	$cv_{1,2}$	L_1	$\pi_{1,2}(L_1)$	$Q_{1,2}$ (95%)
1.00	10	3.693	3.796	1.028	11.079	0.203	11.282
1.25	10	3.046	3.569	1.172	10.014	0.206	10.220
1.50	10	2.583	3.367	1.303	9.152	0.208	9.360
1.75	10	2.233	3.184	1.426	8.421	0.208	8.629
2.00	10	1.957	3.016	1.540	7.781	0.206	7.987
1.00	15	3.936	4.457	1.132	12.632	0.253	12.885
1.25	15	3.289	4.250	1.293	11.582	0.261	11.843
1.50	15	2.826	4.067	1.439	10.726	0.267	10.993
1.75	15	2.476	3.901	1.576	9.992	0.271	10.263
2.00	15	2.200	3.749	1.704	9.346	0.273	9.619
1.00	25	4.073	4.932	1.211	14.518	0.323	14.841
1.25	25	3.425	4.739	1.384	13.456	0.339	13.795
1.50	25	2.963	4.568	1.542	12.573	0.351	12.924
1.75	25	2.613	4.413	1.689	11.805	0.361	12.166
2.00	25	2.337	4.271	1.827	11.120	0.369	11.489

To illustrate the method, it suffices to restrict the attention to the situation $m = 2$, and determine the optimal SL limits and the corresponding quantile SL premiums for the retained risk and the first layer according to Section 4. In our numerical example the collective model of risk theory (2.1) is assumed to have a claim size with mean $\mu = 29'732$ and coefficient of variation $k = 8.507$. The model used for the retained risk is (2.4). The claim number random variable has the mean $\mu_N = 2091.8$ and the coefficient of variation $cv_N = 0.106$. The four parameters of the exponential Pareto model are $\alpha = 490'000$, $\beta = 980'000$, $T = 1'000'000$, $\gamma = 1.65999$. The number of claims above the observation point T is assumed to be Poisson distributed with mean 5.25. To design an optimal risk structure we use the percentile premium principle with $r = 1$. The used quantile levels are $\varepsilon = 80\%$ for the retained risk and $\varepsilon = 95\%$ for the first layer. The given XL deductible structure, the means, standard deviations and coefficients of variation within the layers, as well as the obtained optimal SL limits, the corresponding SL premiums and quantile values are summarized in Table 5.1.

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Werner Hürlimann
Feldstrasse 145
CH-8004 Zürich
Switzerland
E-mail: `whurlimann@bluewin.ch`
URL: `www.geocities.com/hurlimann53`

Summary

For a chain of excess-of-loss reinsurance layers with given deductibles, we determine for each layer a uniquely defined aggregate stop-loss limit. These limits are optimal in the sense that they minimize the expected cost of the total retained risk of the first involved layers under restriction of the corresponding maximum cost to the value-at-risk of the stop-loss risk of the last layer. This result holds provided the price of the stop-loss risk is set according to some specific quantile premium principle. It turns out that the optimal stop-loss limits are equal to the unexpected losses of the stop-loss risks of the different layers under the value-at-risk measure. Analytical approximations of the relevant distributions are used to determine in a fast and numerical stable way the optimal stop-loss limits. A numerical real-life example rounds up the study.

Zusammenfassung

Für eine Kette von Schadenexzedentenrückversicherungen mit gegebenen Selbstbehalten wird für jeden Layer eine eindeutig definierte Stop-Loss Limite bestimmt. Diese Limiten sind optimal in dem Sinne, dass sie die erwarteten Kosten des gesamten Risikos im Eigenbehalt für die ersten involvierten Layer minimieren unter Beschränkung der entsprechenden maximalen Kosten auf den Value-at-Risk des Stop-Loss Risikos des letzten Layers. Dieses Ergebnis ist gültig falls der Preis des Stop-Loss Risikos mit Hilfe eines spezifischen Perzentilprämienprinzips ermittelt wird. Es stellt sich heraus, dass die optimalen Stop-Loss Limiten gleich den unerwarteten Verlusten der Stop-Loss Risiken der verschiedenen Layer für das Value-at-Risk Mass sind. Analytische Approximationen der massgeblichen Verteilungen werden benutzt, um die optimalen Stop-Loss Limiten schnell und numerisch stabil zu ermitteln. Ein reales numerisches Beispiel rundet die Studie ab.

Résumé

Pour une chaîne de réassurance en excess-of-loss avec des franchises données, nous déterminons pour chaque tranche de risque une limite stop-loss unique. Ces limites sont optimales dans le sens qu'elles minimisent les coûts espérés du risque total retenu des premières tranches concernées sous la contrainte que les coûts maximaux correspondants sont limités à la value-at-risk du risque stop-loss de la dernière tranche. Ce résultat est valable pour autant que le prix du risque stop-loss est déterminé à l'aide d'un principe percentile de calcul des primes spécifique. Il s'avère que les limites stop-loss optimales sont égales aux pertes inattendues des risques stop-loss des différentes tranches de risque pour la mesure value-at-risk. Des approximations analytiques des distributions correspondantes sont utilitées pour déterminer rapidement et de façon numériquement stable les limites stop-loss optimales. Un exemple numérique arrondit cette étude.