Note on semi-linear credibility and structural interruption in the Bühlmann-Straub model

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1 Introduction

The origin of semi-linear credibility theory goes back to De Vylder [5]. The basic idea of this credibility estimation technique is not to consider the original data but rather to transform the observed data $X_1, \ldots, X_n$ (e.g. by truncation). The credibility predictor is then applied to the transformed data $Y_1 = f(X_1), \ldots, Y_n = f(X_n)$ with the goal to predict the individual premium $E[X|\Theta]$ for the not transformed claims variable X. A situation with similar consequences is given if we have a structural interruption between the observed claims variables $X_1, \ldots, X_n$ and the future claims variable X for which we want to predict the individual premium $E[X|\Theta]$. In both situations – semi-linear credibility and structural interruption – the conditional moments $E[Y_i|\Theta]$, $\text{Var}(Y_i|\Theta)$, $E[X_i|\Theta]$ and $\text{Var}(X_i|\Theta)$ of the transformed data Y_i and the observed claims variables X_i before structural interruption, respectively, are generally different from the conditional moments $E[X|\Theta]$ and $\text{Var}(X|\Theta)$ of the not transformed claims variable / claims variable after structural interruption. Therefore, in this note we consider a credibility model in which the classical assumptions of the Bühlmann-Straub model [2] only hold for the claims variables $X_1, \ldots, X_n$ and not necessarily also for the claims variable X. This slight extension of the Bühlmann-Straub model allows 1) the derivation of a semi-linear credibility predictor in the Bühlmann-Straub [2] framework and 2) the consideration of a structural interruption between past and future. Finally, we give a recursive premium formula for the semi-linear credibility predictor within the Bühlmann-Straub model.

2 Notation

All of the following is based on the probability space $(\Omega, \mathcal{A}, P)$ and we assume that all random quantities lie in the Hilbert space $L^2(\Omega, \mathcal{A}, P)$ of all square integrable random variables defined on $(\Omega, \mathcal{A}, P)$ with scalar product $\langle Y, Z \rangle = E[Y \cdot Z]$ and norm $\|Y\| = \sqrt{\langle Y, Y \rangle}$.

We consider a portfolio of risks during $n + 1$ periods. The claims behaviour of a risk over all $n + 1$ periods is described by an unobservable parameter θ which is interpreted as the realization of a random variable $\Theta : (\Omega, \mathcal{A}, P) \to \widetilde{\Theta}$. The random variable Θ is known as *risk parameter* and characterizes the structure of the portfolio. For the risk under consideration the random variables $X_1, \ldots, X_n \in L^2(\Omega, \mathcal{A}, P)$ denote the transformed data (semi-linear credibility) or the observed claims variables before structural interruption. The random variable $X \in L^2(\Omega, \mathcal{A}, P)$ denotes the not transformed claims variable (semi-linear credibility) or the claims variable after structural interruption.

A main objective of credibility theory is the derivation of an optimal linear-affine predictor for the future *individual premium* $E[X|\Theta]$ with respect to the quadratic loss. I.e. one is interested in the *credibility predictor* defined by

$$P_n^{Cred} = \underset{\widehat{Z} \in \mathcal{L}_n}{\operatorname{argmin}} E\left[\left(\widehat{Z} - E[X|\Theta]\right)^2\right], \tag{2.1}$$

where

$$\mathcal{L}_n = \left\{ \widehat{Z} \in L^2(\Omega, \mathcal{A}, P) \;\middle|\; \widehat{Z} = \alpha_0 + \sum_{i=1}^{n} \alpha_i \cdot X_i \right\}. \tag{2.2}$$

The credibility predictor can be constructed using Hilbert space theory since it is nothing else but the orthogonal projection of $E[X|\Theta]$ on the linear subspace $\mathcal{L}_n \subseteq L^2(\Omega, \mathcal{A}, P)$. In general the credibility predictor $P_n^{Cred} = \alpha_0 + \sum_{i=1}^{n} \alpha_i \cdot X_i$ can be determined by solving the normal equations

$$\begin{aligned} \alpha_0 &= E[X] - \sum_{i=1}^{n} \alpha_i \cdot E[X_i] \\ \operatorname{Cov}\big(E[X|\Theta], X_j\big) &= \sum_{i=1}^{n} \alpha_i \cdot \operatorname{Cov}(X_i, X_j) \qquad \text{for } j \in \{1, \ldots, n\} \end{aligned} \tag{2.3}$$

(see Bühlmann-Gisler [3], p. 72).

3 Model

The following model is a slight modification of the Bühlmann-Straub model [2].

Model Assumptions 3.1 (Modified Bühlmann-Straub model [2])
The risk is characterized by a risk parameter Θ and we have

a) $X, X_1, \ldots, X_n \in L^2(\Omega, \mathcal{A}, P)$.

b) *Conditionally, given* Θ, *the claims variables* $X_1, \ldots, X_n$ *are independent with*

$$E[X_i|\Theta] = m(\Theta) \tag{3.1}$$

$$Var(X_i|\Theta) = \frac{1}{W_i} \cdot \sigma^2(\Theta) \tag{3.2}$$

for $i = 1, \ldots, n$, *where* $W_1, \ldots, W_n > 0$, $Var\big(m(\Theta)\big) > 0$ *and* $E[\sigma^2(\Theta)] > 0$.

Please note that, unlike to the classical Bühlmann-Straub model in Model 3.1, the conditional independence assumption and the assumptions (3.1)-(3.2) only hold for the transformed data/observed claims variables before structural interruption $X_1, \ldots, X_n$. We believe there are two situations in which such a model could be useful:

1) There is a structural interruption between observed claims variables $X_1, \ldots, X_n$ and the future claims variable X. For instance, such a situation is given if we increase or decrease the existing deductible or sum insured for future claims. In this case the conditional moments $E[X_i|\Theta]$, $\text{Var}(X_i|\Theta)$ of the claims variables before the change of the deductible/sum insured are different from the conditional moments $E[X|\Theta]$, $\text{Var}(X|\Theta)$ of the claims variable after the change of the deductible/sum insured.

2) Every risk is the average of single risks which, given Θ, are i.i.d. I.e. the original observations are given by

$$\frac{1}{W_1} \cdot \sum_{j=1}^{W_1} Z_1^j, \ldots, \frac{1}{W_n} \cdot \sum_{j=1}^{W_n} Z_n^j, \tag{3.3}$$

where $Z_i^j \in L^2(\Omega, \mathcal{A}, P)$, given Θ, are i.i.d. for all $i = 1, \ldots, n$ and $j = 1, \ldots, W_i$. If we transform (e.g. truncate) the random variables Z_i^j by a measurable function $f : \mathbb{R} \to \mathbb{R}$ with $f(Z_i^j) \in L^2(\Omega, \mathcal{A}, P)$, the transformed random variables $f(Z_i^j)$ are also conditionally i.i.d. and we have

$$E[X_i|\Theta] = E\big[f(Z_i^1)|\Theta\big] = m(\Theta) \tag{3.4}$$

$$\text{Var}(X_i|\Theta) = \text{Var}\big(f(Z_i^1)|\Theta\big) = \frac{1}{W_i} \cdot \sigma^2(\Theta), \tag{3.5}$$

where

$$X_1 = \frac{1}{W_1} \cdot \sum_{j=1}^{W_1} f(Z_1^j), \ldots, X_n = \frac{1}{W_n} \cdot \sum_{j=1}^{W_n} f(Z_n^j). \tag{3.6}$$

the transformed data. These data satisfy the Model Assumptions 3.1 and their conditional moments $E[X_i|\Theta]$ and $\mathrm{Var}(X_i|\Theta)$ are generally different from the conditional moments $E[X|\Theta]$ and $\mathrm{Var}(X|\Theta)$ of the not transformed claims variable

$$X = \frac{1}{W_1} \cdot \sum_{j=1}^{W} Z^j. \tag{3.7}$$

The idea of considering transformed data instead of the original ones goes back to De Vylder [5]. De Vylder [5] derives the semi-linear credibility predictor within the model of Bühlmann [1] (i.e. without volume measure). Gisler [4] enhanced this idea resulting in a model which involves a volume measure as model 3.1. In this model the aggregated claims amounts, given the risk parameter Θ, are Poisson distributed and the individual claim sizes are i.i.d. In dealing with the problem of large claims the individual claim sizes then are truncated by a transformation $Y_i = \max(X_i, m)$ and the optimal truncation point m is derived.

4 Results

Theorem 4.1 *Under Model Assumptions 3.1 the credibility predictor is given by*

$$P_n^{Cred} = E[X] + \tau \cdot c_n \cdot \left(\sum_{i=1}^{n} \frac{W_i}{W_\bullet^{(n)}} \cdot X_i - \mu \right), \tag{4.1}$$

where $\mu = E[X_i]$ *for* $i = 1, \ldots, n$ *(past collective premium) and*

$$c_n = \frac{W_\bullet^{(n)}}{\kappa + W_\bullet^{(n)}} \quad \textit{with} \quad \kappa = \frac{E[\sigma^2(\Theta)]}{Var(m(\Theta))} \quad \textit{and} \quad W_\bullet^{(n)} = \sum_{i=1}^{n} W_i \tag{4.2}$$

$$\tau = \frac{Cov(E[X|\Theta], m(\Theta))}{Var(m(\Theta))}. \tag{4.3}$$

Moreover, for the quadratic loss of the credibility predictor holds

$$E\left[\left(P_n^{Cred} - E[X|\Theta] \right)^2 \right] = Var(E[X|\Theta]) - \tau^2 \cdot c_n \cdot Var(m(\Theta)). \tag{4.4}$$

Proof. Using the normal equations (2.3) and $\alpha_\bullet = \sum_{i=1}^n \alpha_i$ we have

$$\tau \cdot \text{Var}\big(m(\Theta)\big) = \alpha_j \cdot \frac{1}{W_j} \cdot E[\sigma^2(\Theta)] + \alpha_\bullet \cdot \text{Var}\big(m(\Theta)\big) \tag{4.5}$$

for $j \in \{1, \ldots, n\}$. This is equivalent to

$$\alpha_j = \frac{\tau}{\kappa} \cdot W_j - \frac{\alpha_\bullet}{\kappa} \cdot W_j \tag{4.6}$$

for $j \in \{1, \ldots, n\}$ and summing up leads to

$$\alpha_\bullet = \frac{\tau}{\kappa} \cdot W_\bullet^{(n)} - \frac{\alpha_\bullet}{\kappa} \cdot W_\bullet^{(n)} = \frac{\tau \cdot W_\bullet^{(n)}}{\kappa + W_\bullet^{(n)}} = c_n \cdot \tau. \tag{4.7}$$

From (4.6) and (4.7) we have

$$\alpha_j = \frac{\tau}{\kappa} \cdot W_j \cdot (1 - c_n) = \tau \cdot c_n \cdot \frac{W_j}{W_\bullet^{(n)}}. \tag{4.8}$$

If we insert this in

$$P_n^{Cred} = E[X] + \sum_{i=1}^n \alpha_i \cdot (X_i - \mu) \tag{4.9}$$

we obtain formula (4.1).
Since $E\big[P_n^{Cred}\big] = E[X]$ and $\text{Cov}\big(P_n^{Cred}, E[X|\Theta]\big) = \text{Var}\big(P_n^{Cred}\big)$ we have for the quadratic loss

$$\begin{aligned}
&E\Big[\big(P_n^{Cred} - E[X|\Theta]\big)^2\Big] \\
&\quad = \text{Var}\big(P_n^{Cred} - E[X|\Theta]\big) \\
&\quad = \text{Var}\big(E[X|\Theta]\big) - \text{Var}\big(P_n^{Cred}\big) \\
&\quad = \text{Var}\big(E[X|\Theta]\big) - \tau^2 \cdot c_n^2 \cdot \text{Var}\Bigg(\sum_{i=1}^n \frac{W_i}{W_\bullet^{(n)}} \cdot X_i\Bigg) \\
&\quad = \text{Var}\big(E[X|\Theta]\big) - \tau^2 \cdot c_n^2 \cdot \Bigg(\frac{E[\sigma^2(\Theta)]}{W_\bullet^{(n)}} + \text{Var}\big(m(\Theta)\big)\Bigg) \\
&\quad = \text{Var}\big(E[X|\Theta]\big) - \tau^2 \cdot c_n \cdot \text{Var}\big(m(\Theta)\big).
\end{aligned} \tag{4.10}$$

This finishes the proof of the theorem. □

Remarks 4.2

- Formulas (4.1) and (4.4) are not new. They coincide with the ones derived in Gisler [4] for a model in which, given Θ, the aggregated claims amounts are Poisson distributed and the individual claim sizes are i.i.d. The equality comes from the fact that the assumptions in the model of Gisler [4] lead to the same conditional moments $E[X_i|\Theta]$ and $\text{Var}(X_i|\Theta)$ as model 3.1 and therefore to the same predictor and quadratic loss. For $W_1 = \cdots = W_n$ formulas (4.1) and (4.4) lead to the semi-linear credibility predictor and its quadratic loss in De Vylder [5] within the model of Bühlmann [1].

- The term in brackets on the right-hand side of (4.1) is the experience adjustment. Since

$$\tau \cdot c_n = \text{Corr}\big(E[X|\Theta], m(\Theta)\big) \cdot \sqrt{\frac{\text{Var}\big(E[X|\Theta]\big)}{\text{Var}\big(m(\Theta)\big)}} \cdot \frac{W_\bullet^{(n)}}{\frac{1}{1/\kappa} + W_\bullet^{(n)}} \tag{4.11}$$

  its influence on the credibility predictor P_n^{Cred} increases with 1) the correlation coefficient between $E[X|\Theta]$ and $m(\Theta)$ 2) the volumes $W_\bullet^{(n)}$ 3) the reciprocal $1/\kappa$ of the credibility coefficient and 4) the quotient of $\text{Var}\big(E[X|\Theta]\big)$ and $\text{Var}\big(m(\Theta)\big)$.

- As in the model of Bühlmann-Straub [2] the semi-linear credibility predictor (4.1) and the quadratic loss (4.4) depend on structural parameters which have to be estimated on the basis of data from the collective. For estimators of the quantities we refer to Bühlmann-Gisler [3].

We can rewrite the credibility predictor (4.1) as follows:

$$P_n^{Cred} = E[X] + \tau \cdot \left(\frac{\kappa}{\kappa + W_\bullet^{(n)}} \cdot \mu + \sum_{i=1}^{n} \frac{W_i}{\kappa + W_\bullet^{(n)}} \cdot X_i\right) - \tau \cdot \mu\,. \tag{4.12}$$

Since the term in brackets on the right-hand side of (4.12) is the credibility predictor $P_n^{Cred,BS}$ for the individual premium $E[X|\Theta]$ within the Bühlmann-Straub model [2] (see e.g. Bühlmann-Gisler [3], Theorem 4.2) it holds

$$P_n^{Cred} = E[X] + \tau \cdot \big(P_n^{Cred,BS} - \mu\big)\,. \tag{4.13}$$

The credibility predictor in the Bühlmann-Straub model [2] can be rewritten recursively,

$$P_n^{Cred,BS} = a_n \cdot X_n + (1 - a_n) \cdot P_{n-1}^{Cred,BS}, \tag{4.14}$$

where

$$a_n = \frac{W_n}{W_\bullet^{(n)} + \kappa} \tag{4.15}$$

(see Bühlmann-Gisler [3], page 221). If we combine the recursive premium formula (4.14) with (4.13) we obtain

$$\begin{aligned} P_n^{Cred} &= E[X] + \tau \cdot \left[a_n \cdot X_n + (1 - a_n) \cdot P_{n-1}^{Cred,BS} - \mu\right] \\ &= E[X] + \tau \cdot \left[a_n \cdot (X_n - \mu) + (1 - a_n) \cdot \left(P_{n-1}^{Cred,BS} - \mu\right)\right] \\ &= E[X] + \tau \cdot a_n \cdot (X_n - \mu) + \tau \cdot (1 - a_n) \cdot \left(P_{n-1}^{Cred} - E[X]\right). \end{aligned} \tag{4.16}$$

This gives the following corollary:

Corollary 4.3 *Under Model Assumptions 3.1 the credibility predictor* (4.1) *can be written recursively as follows:*

$$P_n^{Cred} = E[X] + \tau \cdot (1 - a_n) \cdot \left(P_{n-1}^{Cred} - E[X]\right) + \tau \cdot a_n \cdot (X_n - \mu) \tag{4.17}$$

where

$$a_n = \frac{W_n}{W_\bullet^{(n)} + \kappa}. \tag{4.18}$$

Remarks 4.4

- An application of the recursive premium formula (4.17) is only reasonable in a semi-linear credibility context. In the case of a structural interruption between past and future a recursive premium formula does not make sense per definition.

- The third term on the right-hand side of (4.17) is the experience correction due to the last transformed claims variable / observed claims variable X_n. Its influence on P_n^{Cred} increases with the correlation coefficient between $E[X|\Theta]$ and $m(\Theta)$ (cf. Remarks 4.2) and the volume W_n. The second term on the right-hand side of (4.17) is the experience correction due to the older transformed claims variables / observed claims variables $X_1, \ldots, X_{n-1}$. Its influence also increases with the correlation coefficient between $E[X|\Theta]$ and $m(\Theta)$ but decreases with the volume W_n.

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Abstract

We give a slight extension of the Bühlmann-Straub model [2] which allows 1) the derivation of a semi-linear credibility predictor in the Bühlmann-Straub framework and 2) the consideration of a structural interruption between past and future claims variables.

Zusammenfassung

In dieser Arbeit wird eine Verallgemeinerung des Modells von Bühlmann und Straub [2] vorgestellt, welche 1) die Herleitung eines semi-linearen Credibility-Prädiktor im Bühlmann-Straub-Rahmen gestattet, und 2) die Berücksichtigung eines Strukturbruchs zwischen Vergangenheit und Zukunft im Sinne einer systematischen Veränderung erlaubt.

Résumé

On présente une généralisation du modèle de Bühlmann-Straub [2], qui permet 1) de construire un estimateur de crédibilité semi-linéaire dans le contexte Bühlmann-Straub, 2) de tenir compte d'une rupture systématique de structure entre le passé et le futur.