On the group level Swiss Solvency Test

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1   Introduction

The Swiss Solvency Test (SST) provides a consistent framework both for legal entity and group solvency capital requirements. The underlying reference methodology has recently been outlined in a working paper [2]. This methodology relies on a set of principles which are summarized in stylized form below:

(i)   An insurance group is composed of different legal entities (parent company and subsidiaries) which are potentially supervised by different regulators or unregulated, and a web of legally binding capital and risk transfer (CRT) instruments between these legal entities.

(ii)   The calculation of available and required capital of a regulated entity has to include the CRT instruments and interdependencies with all other group entities.

(iii)   Group diversification effects only exist due to the web of CRT instruments.

(iv)   Subsidiaries can be sold by the parent company at their economic value (available capital) minus some minimum capital requirement[1]. Owning a subsidiary is in this sense a fungible[2] value.

Examples of risk transfer instruments are intra-group retrocession, securitization of future cash flows, guarantees and other contingent capital solutions while capital transfer instruments are for example cash-bonds or dividends.

The above principles go well with the bottom-up[3] framework for group diversification via optimal legally enforceable CRTs that we developed in [4]. The aim of this paper is to elaborate on the common methods in [4] and the group level SST. For this purpose, we formalize the above group level SST principles.

In a first step, a common set of legally binding CRT instruments is identified. The risk management's objective is then to minimize the group capital requirements

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[1]In the SST [2] framework, this is the "market value margin" or "cost of capital".

[2]Fungibility in this context refers to the ability to convert assets into cash or other forms of capital which can be transferred. In general, lack of fungibility has to be taken into account in the SST calculations.

[3]"Bottom-up" here means based on the risk assessments on a legal entity level.

by an optimal choice of the CRT. This leads to a well-posed convex optimization problem. The first order conditions induce a consistent valuation principle (compatible with any prior valuation principle) for CRT instruments.

We then distinguish a particular optimal ("equilibrium") CRT which does not affect the entities' individual available capitals, and which is fair in the sense that no lower than the group level of diversification can overturn the diversification benefit of the entire group. Due to the bottom-up approach, an extra capital allocation step is not necessary. In fact, in the context of the optimized capital and risk structure, the allocated capital is just given by the individual entity's required capital.

It turns out that the definition of the minimum capital requirement in Principle (iv) has a strong impact on the optimal CRT and the respective group diversification effects. Indeed, in a numerical example we show that if the minimum capital requirement is defined as market value margin (or cost of capital), as proposed in [2], then Principle (iv) dominates the effect of any other CRT. In fact, almost the fully consolidated diversification effect is obtained. This raises the question of how to implement, or modify, Principle (iv). We propose that, in any case, the minimum capital requirement in Principle (iv) be distinguished from the market value margin.

The remainder of the paper is as follows. Section 2 contains the formal probabilistic setup for the group capital structure (available capital). In Section 3 we discuss the solvency capital requirement (required capital) from different points of view: stand alone and non diversified, fully consolidated, and SST compatible via CRTs, which formalizes the above group level SST Principles (i)–(iv). Section 4 contains the main results. We characterize optimal CRTs and show how to find them by solving a well-posed convex optimization problem. In Section 5 we illustrate our findings by a concrete example and elaborate on the impact of the minimum capital requirement on the optimal CRT. We conclude by Section 6. The Appendix (Sections A–C) contains some notation and facts from convex analysis and the proofs of our theorems.

2 Group Capital Structure

We consider an insurance group consisting of $m + 1$ legal entities: the parent company (entity 0) and m subsidiaries (entities $1, \ldots, m$). Values at the beginning of the accounting year are deterministic and denoted by small letters. Values at the end of the accounting year are random and denoted by capital letters. We model this randomness, or risk, with the space of integrable random variables L^1

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that all values are already discounted by the prevailing risk free rate.

The current available capital (value of asset-liability portfolio) of entity i is defined as

$$c_i = a_i - \ell_i$$

where a_i and ℓ_i denote the value of assets and best estimate of liabilities, respectively. The terminal value of the asset-liability portfolio of entity i is given as

$$V_i = A_i - F_i - L_i \tag{1}$$

where A_i and L_i denote the terminal value of assets and best estimate of liabilities, respectively, and F_i denotes the claims payments during the accounting year. As in [4] we assume a linear valuation principle $\mathcal{V} : L^1 \to \mathbb{R}$ such that

$$c_i = \mathcal{V}(V_i) \tag{2}$$

for all entities i.

Remark 2.1 In view of Principle (iv), owning the subsidiaries is an asset for the parent company. But to avoid double counting, we assume that a_0 and A_0 are net of the value of owning the subsidiaries. We do, however, take Principle (iv) into account for the realizable distribution of terminal available capital in (7) and (8) below. See also Remark 3.2

3 Required Capital

The SST risk measure is the expected shortfall ES on the confidence level of 99%. The stand alone solvency requirement for entity i is

$$\mathrm{ES}(V_i - mvm_i) \leq 0 \tag{3}$$

where mvm_i denotes the market value margin (or cost of capital, see [1]) that is needed at the end of the accounting year to assure the run-off of the asset-liability portfolio. Put (3) in words: the risk of missing the market value margin ($V_i < mvm_i$) is acceptably low.

The calculation of mvm_i is part of the solo SST. Hence we can assume that mvm_i is a given deterministic parameter. The stand alone required capital (or target capital) for entity i accordingly is

$$k_{\mathrm{stal},i} = \mathrm{ES}(V_i - mvm_i - c_i) = \mathrm{ES}(V_i) + mvm_i + c_i. \tag{4}$$

This results in a non diversified group required capital on a stand alone basis of

$$k_{\text{stal}} = \sum_i k_{\text{stal},i} \, . \tag{5}$$

Remark 3.1 The required capital as an indicator for the risk profile has to be considered with respect to the available capital. Indeed, suppose the available capital is increased by adding assets to its portfolio. In absolute terms, this certainly improves the financial strength for backing the liabilities. And yet, due to the riskiness of the additional assets, the required capital increases too. Hence optimizing the risk profile subject to regulatory requirements amounts to minimize the difference between required and available capital. This approach is taken up below.

3.1 Consolidated View

Under a fully consolidated view (one group balance sheet, assuming full fungibility of capital) the group solvency requirement would be $\text{ES}\left(\sum_i V_i - mvm_i\right) \leq 0$, and the fully diversified group required capital would amount to

$$k_{\text{cons}} = \text{ES}\left(\sum_i V_i - mvm_i - c_i\right) = \text{ES}\left(\sum_i V_i\right) + \sum_i (mvm_i + c_i). \tag{6}$$

In particular, the consolidated group market value margin is given as sum of the respective stand alone margins mvm_i.

Combining (4) and (6), we obtain a (hypothetical) consolidated relative diversification effect of

$$b_{\text{cons}} = 1 - \frac{k_{\text{cons}}}{k_{\text{stal}}} \, .$$

However, this approach is not in line with regulatory practice! According to Principle (iii) of the group level SST, group diversification effects can only be realized via legally binding CRTs.

3.2 CRT View

In view of Principle (iv), the surplus $(V_i - mcr_i)^+$ of subsidiary i exceeding the minimum capital requirement mcr_i is the maximal amount of capital

which is fungible and can be transferred to the parent company. The gross[4] available capitals of the parent company and its subsidiaries, respectively, become

$$C_0 = V_0 + \sum_{i=1}^{m}(V_i - mcr_i)^+ \tag{7}$$

$$C_i = \min\{V_i, mcr_i\}, \quad i = 1, \ldots, m. \tag{8}$$

We assume in the sequel that mcr_i is a deterministic parameter which can be determined according to some SST guidelines.

Remark 3.2 In accordance to (7), the value of owning subsidiary i becomes $\mathcal{V}((V_i - mcr_i)^+)$. In view of Principle (iv), this value could be counted towards the assets of the parent company, a_0. But then, to avoid double counting of capital, a_i would have to be reduced by the same amount, lowering the rating of subsidiary i substantially! This would not be in line with a going concern.
Anyhow, due to the cash-invariance of ES, the group capital requirement does not depend on the initial allocation of available capital. Indeed, the group management may redistribute current available capital across the group in order to increase the rating of its subsidiaries. For example, transferring 100 m euro risk free cash from the parent company to subsidiary 1, increases the subsidiary's available capital by 100 m euro (and reduces the parent company's by 100 m euro) without changing its required capital.

In view of Principle (i), we can assume that a well defined set of CRT instruments exists, with future contingent values modelled by some linearly independent random variables $Z_0, Z_1, \ldots, Z_n$ in L^1. We also assume that cash is fungible between the entities as long as the payments at the end are determined at the beginning of the accounting year. This is expressed by letting $Z_0 \equiv 1$ denote the payoff of a cash-bond.
Formally, a CRT is a matrix $x = (x_i^j)$ in $\mathbb{R}^{(m+1)\times(n+1)}$ satisfying the clearing condition

$$\sum_i x_i^j = 0, \quad j = 0, \ldots, n, \tag{9}$$

which yields the following realizable distribution of available capital across the entities

$$C_{\mathrm{CRT},i} = C_i + \sum_j x_i^j Z_j, \quad i = 0, \ldots, m. \tag{10}$$

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[4]That is, ingnoring the contingent CRT cash flows to be defined below in (10).

The objective of the group (see Remark 3.1) is to minimize the difference between required and available capital, hence to find an optimal CRT $\hat{x}$ which solves the optimization problem

$$\min_{(x_0,\ldots,x_m)} \sum_i \text{ES}\left(C_i + \sum_j x_i^j Z_j\right) \tag{11}$$

subject to the clearing condition (9). The resulting group capital requirement becomes

$$k_{\text{CRT}} = \sum_i k_{\text{CRT},i}$$

with

$$k_{\text{CRT},i} = \text{ES}(C_{\text{CRT},i} - mvm_i - c_i) = \text{ES}(C_{\text{CRT},i}) + mvm_i + c_i. \tag{12}$$

Notice that $k_{\text{CRT},i}$ obtained in (12) can be interpreted as the capital allocated to entity i. Hence, in our framework, we do not need an exogenous capital allocation method. In fact, we will show in Theorem 4.4 below that this capital allocation is fair in some specific sense.

The realizable relative diversification effect becomes

$$b_{\text{CRT}} = 1 - \frac{k_{\text{CRT}}}{k_{\text{stal}}}.$$

Benchmark is the consolidated diversification effect b_{cons}. From theory (sub-additivity of ES) we already know that $b_{\text{CRT}} \leq b_{\text{cons}}$. The goodness of our approach below will be measured by how small we can make the difference $b_{\text{cons}} - b_{\text{CRT}}$.

4   Optimal CRTs

We now formalize the proposed framework and introduce the functions

$$u_i(x) := \text{ES}\left(C_i + \sum_j x^j Z_j\right), \tag{13}$$

for $x = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1}$ and $i = 0, \ldots, m$. Note that u_i is finite-valued since, by assumption, all random variables considered are in L^1.

As a consequence, we can express the constrained $(m + 1) \times (n + 1)$-dimensional optimization problem (11), subject to (9), as follows

$$\min_{\sum_i x_i = 0} \sum_i u_i(x_i) \,. \tag{14}$$

Coherence of ES implies that u_i is convex and "cash-invariant"

$$u_i(x + re_0) = u_i(x) - r \,, \quad \forall r \in \mathbb{R} \,, \tag{15}$$

where $e_0 = (1, 0, \dots, 0), e_1, \dots, e_n$ denotes the standard basis on $\mathbb{R}^{n+1}$. Hence $\partial u_i / \partial x^0 = -1$. To simplify the subsequent discussion[5], we assume that every

$$u_i \text{ is differentiable on } \mathbb{R}^{n+1} \,. \tag{16}$$

Adding long (assets) or short (liabilities) positions in the CRT instruments to the portfolio (10) also changes its current value (available capital). To determine the available and required capital therefore one needs to know the value of adding positions in $Z_0, \dots, Z_n$. We assume that such value is given by a linear indifference valuation principle as follows. Let $x_i \in \mathbb{R}^{n+1}$ represent the portfolio (10) of entity i. We call the linear functional $\mathcal{V} : L^1 \to \mathbb{R}$ an *indifference valuation principle for entity i with respect to x_i* if adding positions $z \in \mathbb{R}^{n+1}$ to x_i is less optimal (that is, requires more capital) than adding the value equivalent cash amount of

$$\sum_j z^j \mathcal{V}(Z_j) = p \cdot z \,,$$

where the value vector $p = p(\mathcal{V}) \in \mathbb{R}^{n+1}$ is defined as $p^j := \mathcal{V}(Z_j)$, and $\cdot$ denotes the scalar product. Formally, this means

$$u_i(x_i + z) \geq u_i(x_i + (p \cdot z)e_0) \quad \forall z \in \mathbb{R}^{n+1} \,. \tag{17}$$

From (17) and the cash-invariance property (15) we derive:

Lemma 4.1 $\mathcal{V}$ *is an indifference valuation principle for entity i with respect to x_i if and only if*

$$p = -\nabla u_i(x_i) \,. \tag{18}$$

In particular, we then have $p \cdot e_0 = p^0 = 1$. Hence the value of a unit of cash is one.

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[5]The following results also hold without this technical assumption, see [4].

Proof Cash-invariance (15) and (17) imply

$$u_i(x_i + z) \geq u_i(x_i) - p \cdot z \quad \forall z \in \mathbb{R}^{n+1}.$$

Hence $-p \in \partial u_i(x_i)$ is a subgradient of u_i at x_i. Since, by assumption (16), u_i is differentiable at x_i, it thus follows that $-p = \nabla u_i(x_i)$, see Section A. The last statement follows again from the cash-invariance (15). $\square$

Consistent valuation across the entities therefore can only take place at CRTs $(x_0, \ldots, x_m)$ where $\nabla u_0(x_0) = \cdots = \nabla u_m(x_m)$. It turns out that this is just the first order condition for the optimization problem (14).

Theorem 4.2 *Let $(\hat{x}_0, \ldots, \hat{x}_m)$ satisfy the clearing condition (9). The following are equivalent:*

(i) $(\hat{x}_0, \ldots, \hat{x}_m)$ *is a minimizer for (14);*

(ii) $\nabla u_0(\hat{x}_0) = \cdots = \nabla u_m(\hat{x}_m)$;

(iii) $(\hat{x}_i^j, i = 1, \ldots, m, j = 1, \ldots, n) \in \mathbb{R}^{m \times n}$ *is a minimizer for the unconstrained $m \times n$-dimensional convex optimization problem*

$$\min_{(x_i^j) \in \mathbb{R}^{m \times n}} \left(u_0 \left(0, -\sum_{i=1}^{m} x_i^1, \ldots, -\sum_{i=1}^{m} x_i^n \right) + \sum_{i=1}^{m} u_i(0, x_i^1, \ldots, x_i^n) \right), \quad (19)$$

and $\hat{x}_0^j = -\sum_{i=1}^{m} \hat{x}_i^j$.

Remark 4.3 Note that a minimizer for (14) is never unique: due to cash-invariance (15), rebalancing the cash $x_i^0 \mapsto x_i^0 + r_i$, for any transfer with $\sum_i r_i = 0$, does not alter the aggregated capital requirements, as

$$\sum_i u_i(x_i) = \sum_i u_i(x_i + r_i e_0).$$

Among the optimal CRTs there is a distinguished one:

Theorem 4.4 *Let $(\hat{x}_0, \ldots, \hat{x}_m)$ be a minimizer for (14), and unambiguously denote $p = -\nabla u_i(\hat{x}_i)$. Then $\overline{x}_i := \hat{x}_i - (p \cdot \hat{x}_i) e_0$ defines an optimal CRT which is also individually optimal in the sense that*

$$p \cdot \overline{x}_i = 0 \quad and \quad \min_{p \cdot z = 0} u_i(z) = u_i(\overline{x}_i), \quad (20)$$

for all entities $i = 0, \ldots, m$. Moreover, it is fair in the sense that

$$\sum_{i \in I} u_i(\overline{x}_i) \leq \min_{\sum_{i \in I} x_i = 0} \sum_{i \in I} u_i(x_i) \tag{21}$$

for every level of diversification $I \subset \{0, \ldots, m\}$.

In view of (21) we may interpret

$$k_{\mathrm{CRT},i} = u_i(\overline{x}_i) + mvm_i + c_i, \quad i = 0, \ldots, m \tag{22}$$

as a fair capital allocation, as announced in Section 3.2 above.

Remark 4.5 In economic theory, the allocation $(\overline{x}_i)$ satisfying (20) is called an *equilibrium*, see [4].

Remark 4.6 Property (20) says in particular that the net value of the equilibrium CRT $\overline{x}_i$ is zero under the valuation principle p for every entity i. Hence it does not affect the current available capital, and is thus distinguished.

Strictly speaking, in order that p be consistent with any prior linear valuation principle (2), we have to assume that V_i does not lie in the linear span of $Z_0, \ldots, Z_n$, for all entities i. Since in this case, we are indeed free to specify $\mathcal{V}(Z_j)$ to be equal to p^j, for all $j = 0, \ldots, n$. This assumption is realistic, as in general the initial asset-liability portfolio of entity i is more diverse than any portfolio consisting solely of the CRT instruments.

Remark 4.7 There is empirical evidence that insurance companies price CRT instruments based on (risk measure) equilibrium valuation principles, such as the present one. Indeed, using date from the U.S. property-liability industry, Cummins et al. [3] provide empirical tests which strongly support the theoretical prediction that prices of illiquid, imperfectly hedgeable intermediated risk products should depend upon firm capital structure, the covariability of the risks with the firms's other projects, and their marginal effects on the firm's insolvency risk.

As for the existence of an optimal CRT, we quote Corollary 7.2 in [4]:

Theorem 4.8 *A minimizer for (14) always exists.*

5   Example

For illustration we consider an insurance group consisting of the parent company and $m = 1$ subsidiary[6]. The current capital structure is

i	a_i	ℓ_i	c_i
0	8	6	2
1	4	3	1.

Hence the parent company is twice the subsidiary in size. For simplicity, we summarize F_i and L_i in one variable, denoted L_i, so that (1) reads $V_i = A_i - L_i$. For $i = 0, 1$, we model A_i normal and L_i log-normal as

$$A_i = a_i \left(1 + \mu + \sigma_A W_A\right)$$
$$L_i = \ell_i \exp \left(\sigma_L W_{L_i} - \sigma_L^2/2\right)$$

with asset return[7] $\mu = 0.01$ and volatility $\sigma_A = 0.02$, log-liability standard deviation $\sigma_L = 0.08$, and $W = (W_A, W_{L_0}, W_{L_1})$ a three dimensional standard[8] normal distributed vector. Hence the asset returns for parent company and subsidiary are perfectly correlated, while their liabilities are independent.

The SST field tests [1] have shown that the market value margin mvm_i ranges between 10% and 60% of the one year risk capital $c_i + \text{ES}(V_i)$. Consistently with these empirical facts, we set

$$mvm_i = 0.4 \times \left(c_i + \text{ES}(V_i)\right), \quad i = 0, 1.$$

The minimum capital requirement mcr_i will vary as a multiple $q_{\text{mcr}} \geq 0$ of the stand alone one year required capital:

$$mcr_i = q_{\text{mcr}} \times \left(c_i + \text{ES}(V_i)\right), \quad i = 0, 1. \tag{23}$$

An interesting indicator is the probability

$$p_{\text{default}} = \mathbb{P}[V_1 < mcr_1]$$

that the subsidiary defaults on the minimum capital requirement, see (7)–(8).

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[6]This can also be interpreted as all subsidiaries summarized by a representative one.

[7]All values are already discounted by the prevailing risk free rate.

[8]That is, its coordinates are mutually independent.

As CRT instrument we use quota share retrocession. The subsidiary can cede a proportion of its liabilities to the parent company, that is, we set

$$Z_0 = 1 \quad \text{and} \quad Z_1 = L_1 \,.$$

The optimal quota follows by minimizing the group required capital (14). Using Matlab with 10^6 sample points for W, we obtain the following stand alone and consolidated numbers, respectively:

$$k_{\text{stal},0} = 1.4 \times 1.3807 = 1.933 \,, \quad k_{\text{stal},1} = 1.4 \times 0.693 = 0.970 \,,$$
$$k_{\text{stal}} = 2.903 \,, \quad k_{\text{cons}} = 2.372 \,, \quad b_{\text{cons}} = 0.183 \,. \tag{24}$$

The CRT figures are determined for varying minimum capital requirement (23), by numerically solving the unconstrained 1-dimensional convex optimization problem (19). The results are as follows:
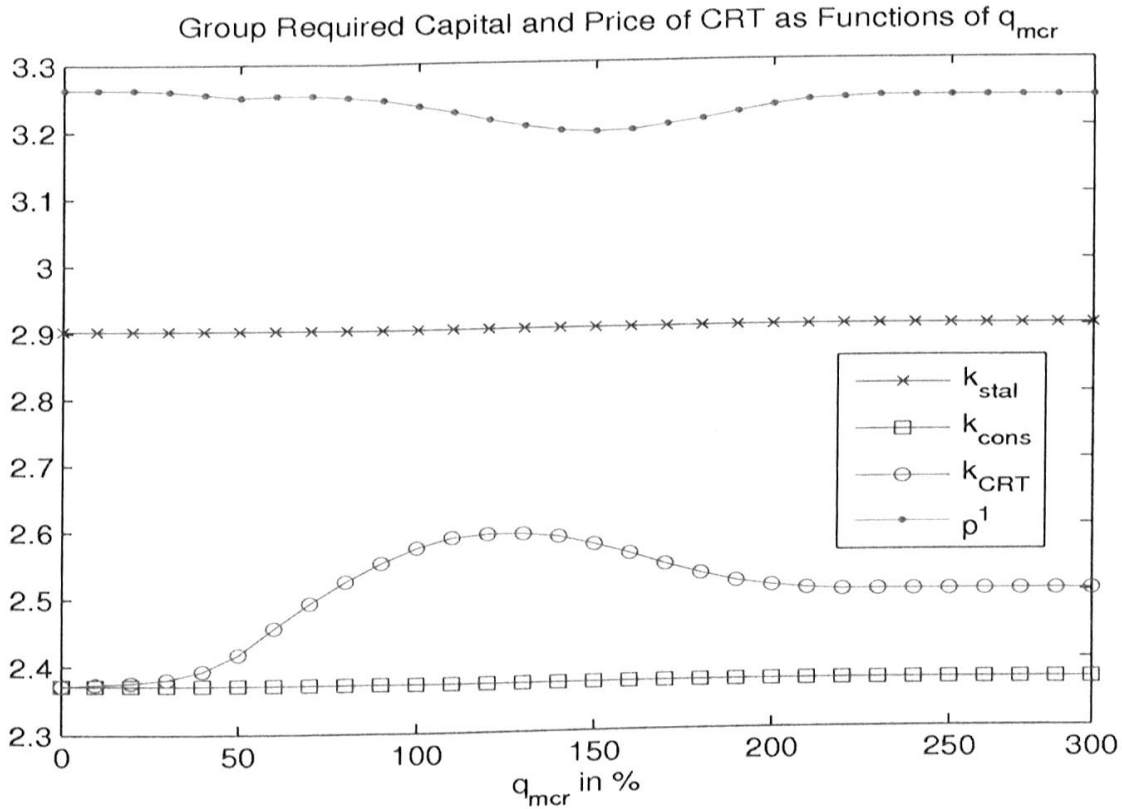

Figure 1: Group required capital k_{CRT} and price p^1 of Z_1 as functions of q_{mcr}.

Figure 1 shows the non monotonic dependence of the group required capital k_{cons} and the equilibrium value p^1 of the CRT instrument $Z_1 = L_1$ on the

minimum capital requirement factor q_{mcr}. The maximal group required capital of 2.594 is attained at $q_{mcr} = 1.2$. On the other hand, for $q_{mcr} \leq 0.4$ we obtain almost the fully consolidated diversification effect, see also Figure 3. The minimal equilibrium value p^1 of 3.19 is attained at $q_{mcr} = 1.5$, which is still greater than the best estimate $\ell_1 = \mathbb{E}[L_1] = 3$ of L_1. The positive difference $p^1 - \ell_1$ equals the risk premium that the subsidiary is willing to pay for ceding a part of its liability risk to the parent company.
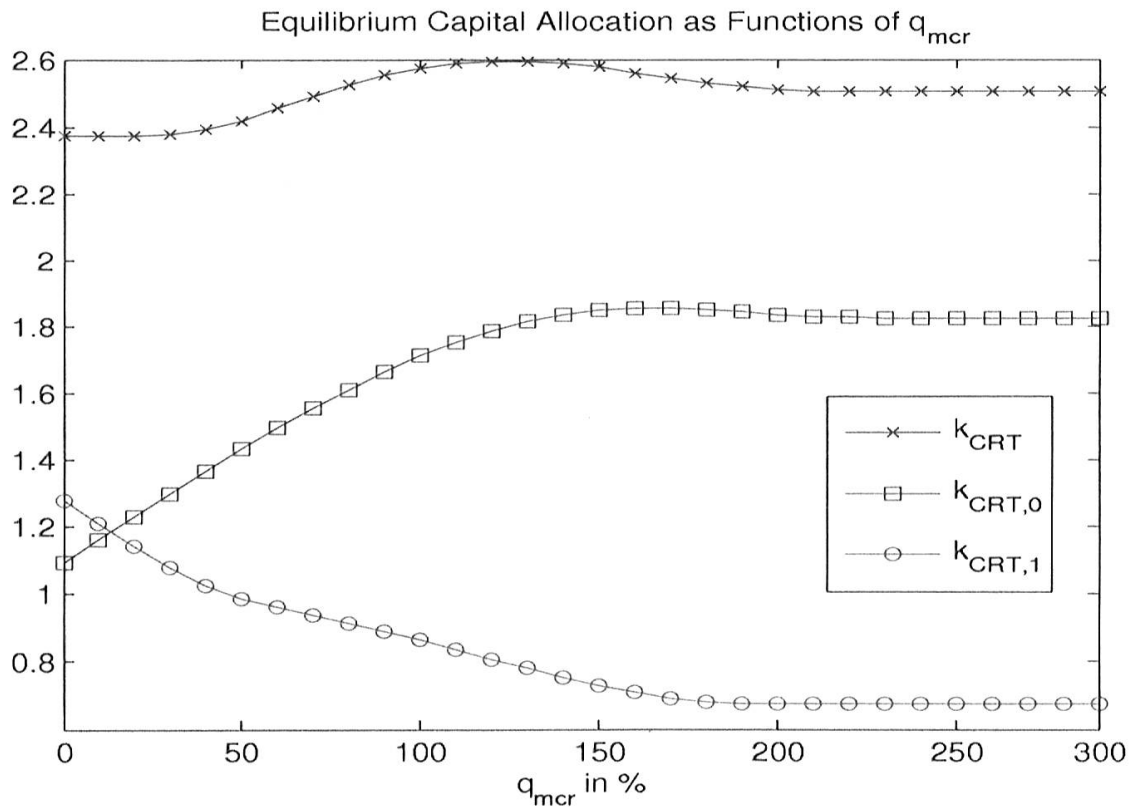

Figure 2: Equilibrium capital allocation $k_{CRT,0} + k_{CRT,1} = k_{CRT}$ as function of q_{mcr}.

Figure 2 shows the dependence of the equilibrium capital allocation $k_{CRT,0} + k_{CRT,1} = k_{CRT}$, see (22), on the minimum capital requirement factor q_{mcr}. The required capital $k_{CRT,0}$ of the parent company attains its maximum of 1.85 at $q_{mcr} = 1.6$.
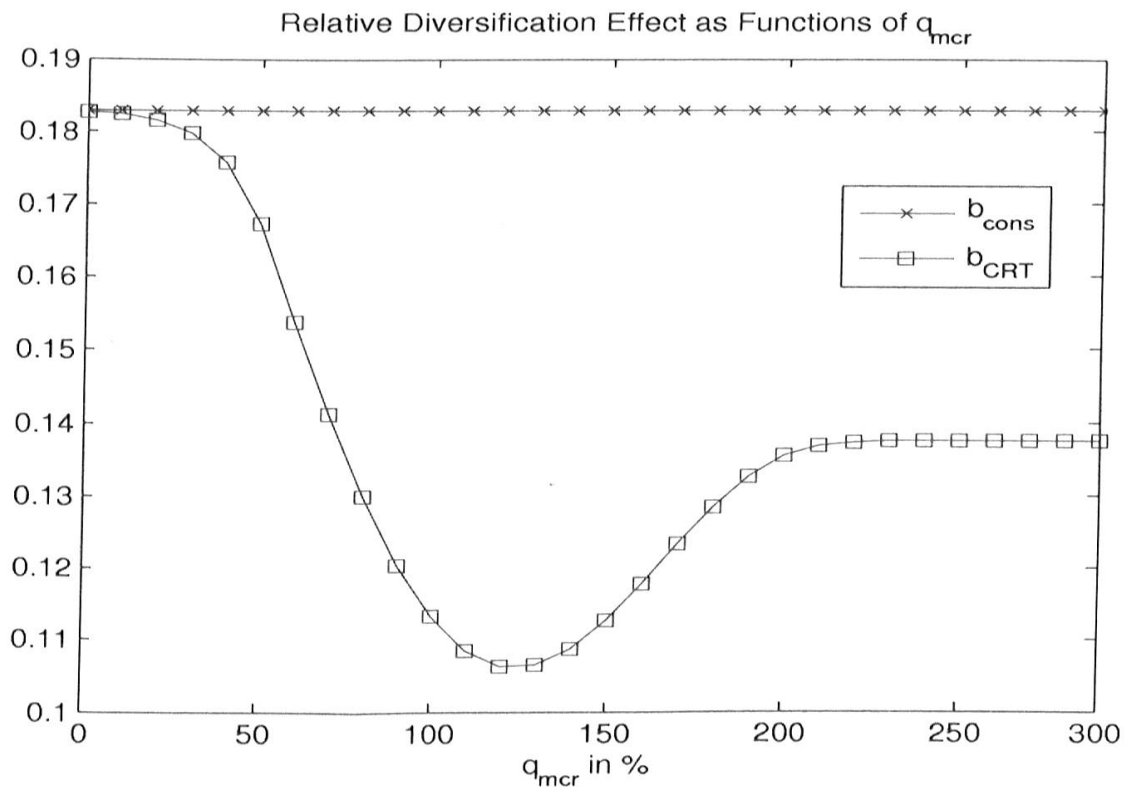
Figure 3: Relative diversification effect b_{CRT} as function of q_{mcr}.

Figure 3 shows the non monotonic dependence of the relative diversification effect b_{CRT} on the minimum capital requirement factor q_{mcr}. The worst relative diversification effect of 0.106 is attained at $q_{\mathrm{mcr}} = 1.2$. For $q_{\mathrm{mcr}} \leq 0.4$, we have $b_{\mathrm{CRT}} \geq 0.180$, hence almost the fully consolidated relative diversification effect (24), which is due to the very small default probability p_{default}, see Figure 4.
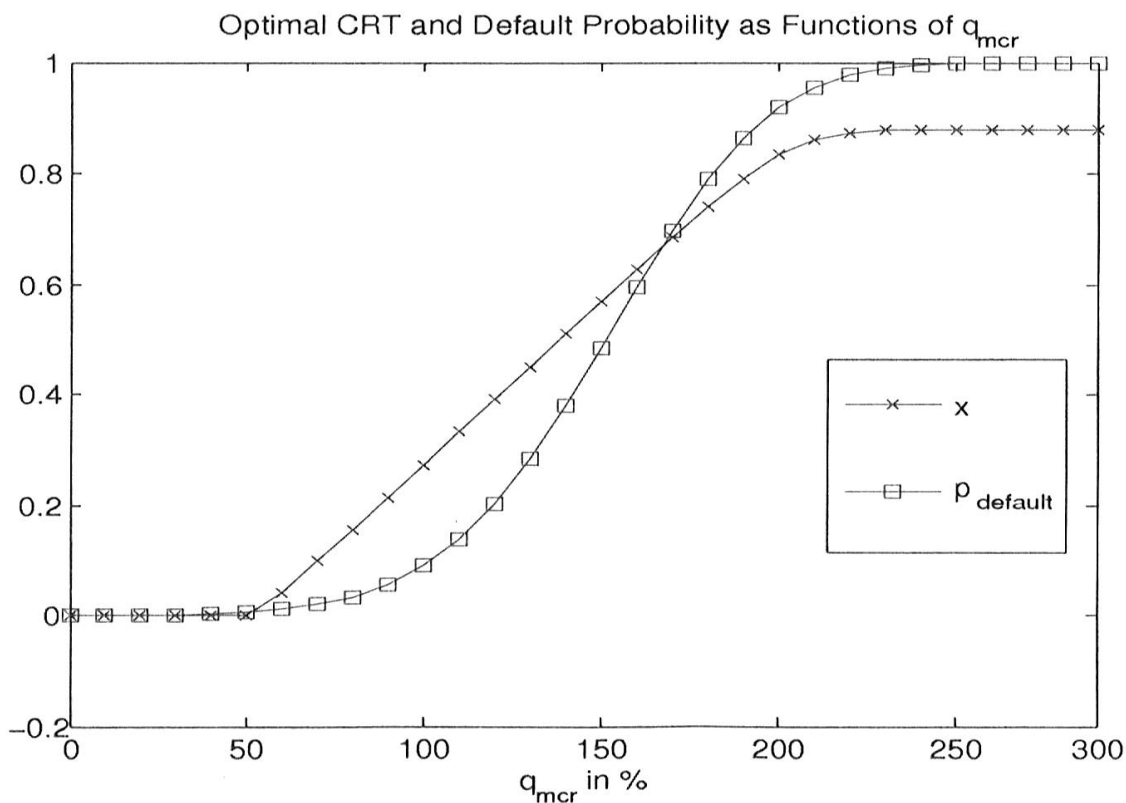
Figure 4: Optimal CRT $\hat{x}_1^1$ (in the figure denoted as "x") and default probability p_{default} as functions of q_{mcr}.

Figure 4 shows the dependence of the optimal CRT $\hat{x}_1^1$ (in the figure denoted by "x") and the default probability p_{default} on the minimum capital requirement factor q_{mcr}. Obviously, both variables are increasing. As for the optimal CRT, we obtain that $\hat{x}_1^1 \to 0.878$ for $q_{\mathrm{mcr}} \to \infty$. On the other hand, we observe that $\hat{x}_1^1 \approx 0$ (up to 5 digits) for $q_{\mathrm{mcr}} \le 0.4$. This is associated with very small default probabilities of $p_{\mathrm{default}} \le 0.003$ for $q_{\mathrm{mcr}} \le 0.4$. Hence the effect of Principle (iv) is essentially equivalent to a fully consolidated view if the minimum capital requirement is defined as (small as) the market value margin. Is this reasonable? We propose that, in any case, the minimum capital requirement in Principle (iv) be distinguished from the market value margin.

Note that omitting Principle (iv) is equivalent to setting $q_{\mathrm{mcr}} = \infty$. From Figure 3 we see that a minimum capital requirement of $q_{\mathrm{mcr}} = 0.8$ yields approximately the same diversification effect via CRTs as if Principle (iv) were omitted.

6   Conclusion

We have formally implemented the stylized principles of the group level SST. We assumed that the risk management's objective is to minimize the group capital requirements by optimizing the web of CRT instruments between the entities. This led to a well-posed convex optimization problem. As byproducts we obtained a consistent valuation principle for the CRT instruments and a fair (equilibrium) capital allocation.

In a numerical example we have elaborated on how the optimal CRT and the respective group diversification effect depend on the minimum capital requirement in SST Principle (iv). It turned out that the effect of Principle (iv) is essentially equivalent to a fully consolidated view if the minimum capital requirement is defined as (small as) the market value margin. This raises the question of how to actually implement, or modify, Principle (iv). In any case, the minimum capital requirement in Principle (iv) should be distinguished from the market value margin. A systematic study is beyond the scope of this paper. We recommend that this aspect be further discussed in the Solvency 2 process.

A   Some Facts from Convex Analysis

The following proofs rely on general principles in convex analysis, which can be found e.g. in [5]. Let $f : \mathbb{R}^d \to (-\infty, +\infty]$ be a lower semi-continuous convex function. Its *conjugate*,

$$f^*(q) := \sup_{x \in \mathbb{R}^d} \left(q \cdot x - f(x) \right),$$

is again a lower semi-continuous convex function $f^* : \mathbb{R}^d \to (-\infty, +\infty]$, and $f^{**} = f$ (see Theorem 12.2 in [5]). The *effective domain* of f is defined as

$$\mathrm{dom}(f) = \{ q \mid f(q) < \infty \} .$$

The *subgradients* of f form a (possibly empty) convex set

$$\partial f(x) = \left\{ q \in \mathbb{R}^d \mid f(x + z) \geq f(x) + q \cdot z \ \forall z \in \mathbb{R}^d \right\} ,$$

and are characterized by

$$q \in \partial f(x) \quad \Leftrightarrow \quad f(x) + f^*(q) = q \cdot x , \tag{25}$$

see Theorem 23.5 in [5]. Furthermore, $\partial f(x)$ consists of a single element if and only if f is differentiable at x. In this case $\partial f(x) = \{\nabla f(x)\}$, where ∇f denotes the gradient of f, see Theorem 25.1 in [5].

B Proof of Theorem 4.2

It follows by the finiteness of u_i and Theorem 5.4 in [5] that

$$u(y) := \inf_{y=\sum_{i=0}^{m} x_i} \sum_{i=0}^{m} u_i(x_i) \tag{26}$$

defines a convex function $u : \mathbb{R}^{n+1} \to [-\infty, +\infty)$. Theorem 16.4 in [5] states that its conjugate satisfies

$$u^*(q) = \sum_{i=0}^{m} u_i^*(q) \quad \forall q \in \mathbb{R}^{n+1} . \tag{27}$$

Obviously, the constrained optimization problem (14) is equivalent to (26) for $y = 0$.

(i) $\Rightarrow$ (ii). By assumption, all random variables considered are in L^1 and thus $u(0) = \sum_{i=0}^{m} u_i(\hat{x}_i) > -\infty$, see (13). It follows from Theorem 7.2 in [5] that u is finite-valued on $\mathbb{R}^{n+1}$. In view of Theorem 23.4 in [5], there exists a subgradient $q \in \partial u(0)$. Using (25) and (27) and the clearing condition, we conclude that

$$u(0) = -u^*(q) = \sum_{i=0}^{m} -u_i^*(q) = \sum_{i=0}^{m} (q \cdot \hat{x}_i - u_i^*(q)) \leq \sum_{i=0}^{m} u_i(\hat{x}_i) = u(0) .$$

Therefore $q \cdot \hat{x}_i - u_i^*(q) = u_i(\hat{x}_i)$ and thus $q = \nabla u_i(\hat{x}_i)$, see (25), for all i.

(ii) $\Rightarrow$ (i). Write $q = \nabla u_i(\hat{x}_i)$. In view of (25) and (27), it follows that

$$u(0) \geq -u^*(q) = \sum_{i=0}^{m} (q \cdot \hat{x}_i - u_i^*(q)) = \sum_{i=0}^{m} u_i(\hat{x}_i) .$$

Hence $(\hat{x}_0, \ldots, \hat{x}_m)$ is a minimizer for (14).

(i) $\Leftrightarrow$ (iii). This follows from the cash-invariance (15).

C Proof of Theorem 4.4

We have, by the properties of u_i and p,

$$\inf_{p \cdot y = 0} u_i(y) = \inf_y u_i(y - (p \cdot y)e_0)$$

$$= \inf_y (u_i(y) + p \cdot y)$$

$$= -\sup_y (-p \cdot y - u_i(y))$$

$$= -u_i^*(-p),$$

and on the other hand,

$$u_i(\overline{x}_i) = u_i(\hat{x}_i) + p \cdot \hat{x}_i = -p \cdot \hat{x}_i - u_i^*(-p) + p \cdot \hat{x}_i = -u_i^*(-p). \qquad (28)$$

This proves (20). Using the clearing condition and (28) again, we obtain

$$\sum_{i \in I} u_i(\overline{x}_i) = \sum_{i \in I} -u_i^*(-p)$$

$$= \inf_{\sum_{i \in I} x_i = 0} \sum_{i \in I} (-p \cdot x_i - u_i^*(-p))$$

$$\leq \inf_{\sum_{i \in I} x_i = 0} \sum_{i \in I} u_i(x_i),$$

and the theorem is proved.

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Abstract

In this paper we elaborate on Swiss Solvency Test (SST) consistent group diversification effects via optimizing the web of capital and risk transfer (CRT) instruments between the legal entities. A group level SST principle states that subsidiaries can be sold by the parent company at their economic value minus some minimum capital requirement. In a numerical example we examine the dependence of the optimal CRT on this minimum capital requirement. Our findings raise the question of how to actually implement this group level SST principle and how to define the respective level of minimum capital requirements, in particular.

Zusammenfassung

In diesem Beitrag stellen wir Swiss Solvency Test (SST) -konforme Gruppen-Diversifikationseffekte vor, durch Optimieren des Netzes von Kapital- und Risikotransfer-(CRT) Instrumenten zwischen den juristischen Einheiten der Gruppe. Ein Prinzip für den gruppenweiten SST besagt, dass die Muttergesellschaft ihre Tochtergesellschaften zum Marktwert, abzüglich gewisser minimaler Kapitalanforderungen, verkaufen kann. In einem numerischen Beispiel untersuchen wir die Abhängigkeit des optimalen CRT von dieser Minimal-Kapitalanforderung. Unsere Ergebnisse führen zur Frage, wie man dieses gruppenweite SST-Prinzip tatsächlich durchführen soll, und im Speziellen, wie das jeweilige Niveau der minimalen Kapitalanforderung definiert werden kann.

Résumé

Dans cette contribution nous présentons les effets de diversification d'un groupe conformément au test suisse de solvabilité (SST) par l'optimalisation des réseaux d'instruments de transfert des capitaux et des risques (CRT) entre les entités légales d'un groupe. Selon un principe du SST d'un groupe la compagnie mère peut vendre une de ses filiales à sa valeur de marché amputée de certaines exigences en capital minimales. Dans un exemple numérique nous étudions comment le CRT optimal dépend de l'exigence en capital minimum. Nos résultats nous amènent à poser la question de l'implémentation concrète de ce principe du SST d'un groupe et de la définition du niveau des exigences minimales en capitaux.