Valuation of guaranteed unit linked contracts

B. Wissenschaftliche Mitteilungen

Alessandro Jori, Martigny

Valuation of guaranteed unit linked contracts

1 Introduction

The Swiss Solvency Test requires that insurance companies use market-valuation methods in assessing their portfolios of policies. This requirement comes from the need to value coherently the asset and the liability sides of the balance sheet. Furthermore, all risk factors having a material impact on an insurance company capacity to fulfill its obligations should be incorporated in the underlying model of these methods.
In the case of assets, their values are given by "the market." This assumes that the insurance company owns assets that can be traded in a "liquid" market.
In the case of liabilities, the cash flows generated by the insurance contracts should be replicated by financial instruments that can be traded in a market. However this is not the case for the majority of insurance contracts. When we say a "market consistent value" method we mean a valuation method that uses the same principles that govern the valuation of financial instruments that are traded in a financial market; especially important is the assumption that the market is auto-regulating, making arbitrage opportunities inexistent, or at least very hard to find (see [3]). Difficulties do not arise at the conceptual level: insurance contracts are indeed material and generate over time cash flows that can be expressed in units of basic financial instruments. The difficulty consists in selecting the best model that, once calibrated, provides a good representation of what is observed in the market.
In this paper, we study the valuation of a unit linked insurance policy with periodic premiums and a guaranteed minimal amount in case of death or at maturity (or both). This type of contracts is characterized by the fact that their benefit level is uncertain and that this uncertainty is both financial and biometric in nature. Traditional endowment insurance and also life insurance with only death benefits can, in fact, be seen as special cases of the policy we consider here. The literature on the subject is abundant (see for example [1], [2], and the references therein). We will focus on three aspects of the problem that are the necessary steps to value any type of contracts. We consider them in their natural time sequence.
The first aspect is the modeling of a template policy. This is done through the description of the cash flows it generates. This is the simplest task, among the

three, but certainly the most important. The approach we will use can, in principle, be applied to all types of policies.
The second aspect of the problem is the evaluation of the cash flows. We assume that insurance risk is diversified in a large portfolio; the evaluation of the insurance risk is based on the real probabilities, or best estimate probabilities, of the different outcomes that generate the cash flows. On the other hand, the evaluation of financial risks is done under the assumption that the financial market is complete. The value, at a given time t, of a (stochastic) cash flow at time $t+n$ is the same as the value of a portfolio composed of basic financial instruments that replicate, after n time units, all the possible outcomes for that cash flow. The considered cash flows can be outgoing (benefit) as well as incoming (premium). The policy's market-value (or market consistent value) at time t is then the sum of the values of all those portfolios needed to replicate the future cash flows. We will show that under very general hypotheses on the market evolution (i.e., on interest rates, and investment fund values) the market-consistent value of the policy, at any instant, can simply be expressed as the sum of three components:

a) the market value of the fund,

b) the value of the guarantee at maturity,

c) the value of the remaining outgoing cash flows that consist in the value of the "risk plus cost plus surrender" process.

This result, although intuitive, requires some work to derive it rigorously.
The last issue that has to be addressed is the practical approach to be used. In other words, what methods should be utilized for a real world case. In practice, two questions have to be answered: first, which market models can be used? Second, what calculation methods that are simple, stable and fast enough can be used to analyze large portfolios in a reasonable amount of time. Our answer to the first question is to use a very simple model where interest rates (bond market) are deterministic and the investment funds considered are modeled by the simplest diffusion model. Even in this simplified framework, the valuation of a real world case is far from trivial: the guaranteed minimal amount in case of death or at maturity corresponds to Asian options. The level of these benefits does not only depend on the value of one unit of the fund at the moment the benefit becomes payable but also on the path that has been followed to reach that level of the fund's unit-value. This is due to the investment of the premiums. Numerical simulations or analytical approximation methods are needed to value that kind of options.
There are several minor elements of the equity-based contracts that we do not attempt to model. Among them, there is the "cash value option" that allows the insured to cancel the policy and entitles him to cashing an amount of

money whose level is described in the contract. This right has indeed a value. However, it is difficult to describe and model the factors that will lead to the cancellation of the contract by the insured. We think that external factors (taxation privileges, personal reasons, desire to remain insured, bad advices ...) have such an importance that it is unrealistic to believe that we can reasonably model the relation that exists between the decision to cancel the policy and the evolution of the fund. Therefore, we decided to use deterministic withdrawal rates, based only on statistical data coming from "a global portfolio".

2 Notation and assumptions

The insurance risk is modeled with a probability space $(\Omega_1, \Im^1, P)$ and a family of random variables $\{Z(t);\, t \in \mathbb{N}\}$ with values in $\{"A", "O", "D"\}$. Time $t = 0$ is the valuation date ("today") and the basic time unit can be a month, a year, etc. The random variable $Z(t)$ represents the state at time t of the insured initially aged x. We assume that this insured is taken from a portfolio of "active" contracts at time 0. Consequently, $Z(t)$ is defined by

$$Z(t) = \begin{cases} "A" & \text{if the insured is active at time } t, \\ "O" & \text{if the insured has withdrawn at time } t, \\ "D" & \text{if the insured is dead at time } t. \end{cases}$$

Thus, $Z(0) \equiv "A"$ by assumption.
The process $\{Z(t);\, t \in \mathbb{N}\}$ is supposed to be Markovian. The probability measure P is defined via the stochastic transition matrices from time t to $t+1$

$$m(t) = \begin{bmatrix} (1-q_t)(1-h_t) & (1-q_t)h_t & q_t \\ 0 & (1-q_t) & q_t \\ 0 & 0 & 1 \end{bmatrix}$$

The positive functions $0 \leq h_t,\ q_t \leq 1$ are the withdrawal and the death probabilities for the time period $(t, t+1)$ (depending also on other parameters like age, sex, contract duration, time elapsed since issuance, etc.). For simplicity, we assume that lapses occur immediately before time $t+1$, i.e. surrender values are paid at the end of the period.
The transition matrix from t to s, $s \geq t$ is given by

$$M(t, s) = \prod_{l=t}^{s-1} m(l)$$

with $M(t, t) = 1$.

To simplify the forthcoming mathematical expressions we define three events

$$A(t) = \{\omega \mid Z_\omega(t) = A; Z_\omega(t-1) = A; \ldots; Z_\omega(1) = A\},$$
$$O(t) = \{\omega \mid Z_\omega(t) = O; Z_\omega(t-1) = A; \ldots; Z_\omega(1) = A\},$$
$$D(t) = \{\omega \mid Z_\omega(t) = D; Z_\omega(t-1) = A; \ldots; Z_\omega(1) = A\},$$

and, for $s \geq t$,

$${}_tp_s = M(t,s)_{1,1}$$

the probability that the insured is in state "A" at time s given he was in that state at time t.
To describe the information up to time t, we use the filtration $\mathbb{F}^1 = \{\Im_t^1\}_{t\geq 0}$ where $\Im_t^1$ is the σ-algebra generated by $\{Z(s);\ s \leq t\}$.
The financial risks, or market risks, are modeled using a probability space $(\Omega_2, \Im^2, P_2)$, a filtration $\mathbb{F}^2 = \{\Im_t^2\}_{t\geq 0}$, and the (adapted) processes $\{F(t);\ t \in \mathbb{R}^+\}$ and $\{r(t);\ t \in \mathbb{R}^+\}$; the first process represents the market value of one unit of the fund. Premiums paid by the insureds are invested in that fund. We set $F(0) = 1$ by convention.
The second process represents the (instantaneous) short rate at time t. We choose as numeraire

$$\beta(t) = e^{\int_0^t r(s)\,ds}$$

the money market (bank account) that grows at short rate $r(t)$.
The probability measure P_2 represents the "physical" measure that describes the statistical behavior of $F(t)$ and $r(t)$. We assume that there exists a unique equivalent martingale measure Q_2 such that the process $\{F(t)\,/\,\beta(t);\ t \in \mathbb{R}^+\}$ is a $(\Im^2, Q_2)$ martingale. This implies that the financial market is complete.
The value at time t of a unit of money at time s, $s \geq t$, is given by the zero-coupon bond with price

$$P(t,s) = \beta(t)\ E_{Q_2}\left(\frac{1}{\beta(s)}\,\middle|\,\Im_t^2\right).$$

In the rest of the paper, except otherwise stated, time is assumed to be discrete, i.e. we observe the processes $r(t)$ and $F(t)$ only at discrete times.
The probabilistic setting is thus a probability space $(\Omega, \Im) = (\Omega_1 \times \Omega_2,\ \Im_1 \vee \Im_2)$, a filtration $\mathbb{F} = \{\Im_t^1 \vee \Im_t^2\}_{t\in\mathbb{N}}$, and three processes $Z(t)$, $F(t)$, and $r(t)$, $t \in \mathbb{N}$.

3 The contract

We denote by $S(0)$ the value of the assets in the fund at the valuation date, and by $S(t)$ the value of the assets in the fund at time t (after investing the premiums and deducting the management fees proportional to the assets but before the deposit of the premium for the period of time $(t, t+1])$. Finally, we denote by $T \in \mathbb{N}$, $T > 0$, the time at which the policy expires.

Let $P_{sa}(t)$ be the saving premium invested in the fund at time t (for period $(t, t+1]$), P_{tot} be the total premium. The quantity $P_r(t) = P_{tot} - P_{sa}(t)$ is then the premium for death and withdrawal benefits and expenses for that period, and $C(t)$ denotes the annual cost .

We make the following assumptions:

a. premiums are paid at the beginning of the period,

b. benefits (including expenses) are paid at the end of the period,

c. saving premiums are immediately invested in the fund,

d. saving premiums are independent of the value of the fund (which means that the insured knows in advance the part of the premium that will be invested in the fund),

e. expenses, for $t > 0$, are expressed as

$$C(t) = K(t) + \bar{\delta}\left(S(t-1) + P_{sa}(t-1)\right) \frac{F(t)}{F(t-1)}$$

with a deterministic positive function $K(t)$. The factor $\bar{\delta} \geq 0$ corresponds to the management fee proportional to the fund value. To simplify the formulas presented below, we write $\delta = -\ln(1-\bar{\delta})$;

f. surrender values, for $t = 1, \dots, T$, amount to $SV(t) \equiv S(t) - EZ(t)$, where EZ is a deterministic function.

At time t, $t = 1, \dots, T$, we then have

$$S(t) = \frac{F(t)}{F(t-1)} \left(S(t-1) + P_{sa}(t-1)\right)\left(1 - \bar{\delta}\right),$$

and consequently

$$S(t) = S(0)\, F(t)\, e^{-\delta t} + \sum_{u=0}^{t-1} P_{sa}(u)\, e^{-\delta(t-u)} \frac{F(t)}{F(u)}.$$

We finally denote by G_D the minimum insured capital payable in case of death during the period of time $(0, T)$ and by G_L the minimum amount of the pure endowment payable at time T if the insured survives.
The cash flows can be decomposed in several components:

1. maturity benefit payable at time $t = T$;
2. withdrawal benefits payable at time $t = 1, \ldots, T$;
3. death benefits payable at time $t = 1, \ldots, T$;
4. expenses payable at time $t = 1, \ldots, T$;
5. premiums to be received by the insurer at time $t = 0, \ldots, T - 1$.

Then, the first step consists in specifying these different "cash flows" that define in a precise way the insurance policy (contract). Denoting by I_B the indicator function of an event B, we can express all the relevant quantities:

1. Maturity benefit

The maturity benefit at time t is given by

$$\begin{aligned} CF^L(t) &= I_{\{t=T\}} I_{A(t)} \max(G_L; S(t)) \\ &= I_{\{t=T\}} I_{A(t)} S(t) + I_{\{t=T\}} I_{A(t)} (G_L - S(t))_+ \end{aligned}$$

i.e., the maturity benefit corresponds to the value of the shares in the fund plus an Asian put option (the value of the guarantee).

2. Withdrawal benefit

The withdrawal benefit at time t, if withdrawal occurs in $(t-1, t]$, is given by

$$CF^O(t) = I_{\{t>0\}} I_{O(t)} (S(t) - EZ(t))$$

4. Death benefit

The death benefit at time t, if death occurs in $(t-1, t]$, is given by

$$\begin{aligned} CF^D(t) &= I_{\{t>0\}} I_{D(t)} \max(G_D; S(t)) \\ &= I_{\{t>0\}} I_{D(t)} S(t) + I_{\{t>0\}} I_{D(t)} (G_D - S(t))_+ \end{aligned}$$

i.e., the death benefit corresponds to the value of the shares in the fund plus an Asian put option (the value of the guarantee).

4. Expenses

Expenses can be expressed as

$$CF^{C}(t) = I_{\{t>0\}} I_{A(t-1)} \left(\left(e^{\delta} - 1 \right) S(t) + K(t) \right)$$

5. Premiums payable (while the insured is active)

The premiums are given by

$$\begin{aligned} CF^{P}(t) &= CF^{P_{sa}}(t) + CF^{P_{r}}(t) \\ &= I_{\{t<T\}} I_{A(t)} P_{sa}(t) + I_{\{t<T\}} I_{A(t)} P_{r}(t) \end{aligned}$$

for $t = 0, \ldots, T-1$.
Finally, the global cash flows generated by the contract are defined by

$$\begin{aligned} CF^{tot}(t) = {} & CF^{L}(t) + CF^{O}(t) + CF^{D}(t) + CF^{C}(t) \\ & - CF^{P_{sa}}(t) - CF^{P_{r}}(t) \, . \end{aligned}$$

4 Market consistent value

Defining a market consistent value (or simply "market value") at time t for this contract consists in determining a value for each of the preceding cash flows.

Definition Let $Q = P \times Q_2$; the market value at time $0 \leq t \leq T$ is given by the random variable

$$MV(t) = \beta(t) E_{Q} \left(\sum_{l=t}^{T} \beta(l)^{-1} CF^{tot}(l) \middle| \Im_{t} \right) .$$

It can be shown that this value can be written as follows (see the Appendix for the proof)

$$MV(t) = I_{A(t)} \left(S(t) + MV_{1}(t) + MV_{2}(t) \right) \tag{4.1}$$

with

$$MV_{1}(t) = {}_{t}p_{T} \, \beta(t) \; E_{Q_2} \left(\beta^{-1}(T) \left(G_{L} - S(T) \right)_{+} \middle| \Im_{t}^{2} \right) \tag{4.2}$$

which is the value of the guarantee at maturity, and

$$\begin{aligned} MV_2(t) = & \sum_{l=t+1}^{T} {}_t p_{l-1}\, q_{l-1}\, \beta(t)\, E_{Q_2}\left(\beta^{-1}(l)\,(G_D - S(l))_+ \mid \Im_t^2\right) \\ & - \sum_{l=t}^{T-1} {}_t p_l\, P(t,l)\, P_r(l) \\ & + \sum_{l=t+1}^{T} {}_t p_{l-1}\, P(t,l)\, K(l) \\ & - \sum_{l=t+1}^{T} {}_t p_{l-1}\, (1 - q_{l-1})\, h_{l-1}\, P(t,l)\, EZ(l)\,. \end{aligned} \tag{4.3}$$

which is the value of the "risk plus cost plus surrender" process.

Remarks

1. In the approach presented above, we assumed that the functions P_r, K, and EZ are deterministic; in principle, it is sufficient to consider them as adapted processes (i.e. $\mathbb{F}$-measurable). The only change required consists in replacing the expressions $P(t,l)\,f(l)$, $f(l) = P_r(l)$, $K(l)$ or $EZ(l)$, by $\beta(t)\, E_{Q_2}\left(\beta^{-1}(l)\, f(l) \mid \Im_t^2\right)$.
2. Under quite general conditions on the dynamics and the regularity of the processes $r(t)$, and $F(t)$, it is possible to perform a change of variables by choosing as numeraire the zero-coupons. In that case, the expression $\beta(t)\, E_{Q_2}\left(\beta^{-1}(l)\,(G - S(l))_+ \mid \Im_t^2\right)$ takes the more well-known form $P(t,l)\, E_{Q_2^T}\left((G - S(l))_+ \mid \Im_t^2\right)$.

   The probability measure Q_2^T, called the "forward risk adjusted measure," takes into account the market price for the risk inherent to the rates. Such an approach is used in [1]. For deterministic short rate processes, both measures Q_2 and Q_2^T are identical.
3. The Risk Bearing Capital in the Swiss Solvency Test is defined as the difference between the market value of assets and the market consistent value of the policy. If no hedging strategy is in place and the savings premiums are directly invested in the fund, we have that the RBC is just (with opposite sign and without a "market value margin") the value of the "risk plus cost plus surrender" process plus the value of the guarantee at the end of the contract.

4. *Call-Put parity*

If we use the identity $\max(x-y,0) = x-y+\max(y-x,0)$, for $x, y \in \mathbb{R}$, and define, for $t = 0, \ldots, T-1$, $l = t+1, \ldots, T$, an $\mathbb{F}$ adapted process $SV(l)$ such that

$$EZ(l) = S(l) - SV(l)\,,$$

$MV(t)$ can be rewritten as

$$\begin{aligned}
MV(t) = I_{A(t)}\Big\{ & {}_tp_T\, P(t,T)\, G_L \\
& + \sum_{l=t+1}^{T} \Big[P(t,l)\, {}_tp_{l-1}\, q_{l-1}\, G_D \\
& + {}_tp_{l-1}(1-q_{l-1})\, h_{l-1}\, \beta(t)\, E_{Q_2}\left(\beta^{-1}(l)\, SV(l) \,|\, \Im_t^2\right) \Big] \\
& - P_{tot} \sum_{l=t}^{T-1} {}_tp_l\, P(t,l) \\
& + \sum_{l=t+1}^{T} {}_tp_{l-1}\, P(t,l)\, K(l) \\
& + \sum_{l=t+1}^{T} \Big[{}_tp_{l-1}\, q_{l-1}\, \beta(t)\, E_{Q_2}\left(\beta^{-1}(l)\, (S(l) - G_D)_+ \,|\, \Im_t^2\right) \\
& + {}_tp_T\, \beta(t)\, E_{Q_2}\left(\beta^{-1}(l)\, (S(T) - G_L)_+ \,|\, \Im_t^2\right) \Big] \Big\}\,.
\end{aligned}$$

When $G = G_D = G_L$, the first five lines in the above expression give the market consistent value of a classical endowment contract with surrender value given by $SV(l)$, $l = t+1, \ldots, T$. The last two lines are a valuation of the "excesses" paid in case of death or maturity, assuming that the "assets" (in which saving premiums are invested) have the same behavior as $F(t)$ and that 100% of the "excesses" are given back to the insureds.

With $G_L = 0$, we obtain the market consistent value of a classical temporary life insurance contract.

5 Model selection

At the valuation date ($t = 0$), we have to determine the value of $MV_1(0) + MV_2(0)$, $S(0)$ being given by the market. In the rest of this paper, we decide

to work with a deterministic short rate; this assumption is not restrictive in our context and it simplifies the analyses. In [1], one finds a general model for the market. That model takes into account the uncertainty of the evolution of interest rates and their interaction with the evolution of the fund.
Thus, we have to value the value of the guarantee at maturity

$$MV_1(0) = {}_0p_T \, P(0,T) \; E_{Q_2}\left((G_L - S(T))_+\right)$$

and the other benefits (death and withdrawal) and expenses

$$\begin{aligned} MV_2(0) = & \sum_{l=1}^{T} {}_0p_{l-1} \, q_{l-1} \, P(0,l) \; E_{Q_2}\left((G_D - S(l))_+\right) \\ & - \sum_{l=0}^{T-1} {}_0p_l \, P(0,l) \; P_r(l) \\ & + \sum_{l=1}^{T} {}_0p_{l-1} \, P(0,l) \; K(l) \\ & - \sum_{l=1}^{T} {}_0p_{l-1} \, (1 - q_{l-1}) \; h_{l-1} \, P(0,l) \; EZ(l) \end{aligned}$$

The only difficulty with these last expressions consists in determining the value of terms of the form

$$E_{Q_2}\left((G - S(l))_+\right)$$

for $l = 1, \ldots, T$, and $G \in \mathbb{R}$.
Therefore, we need to specify the risk-neutral measure Q_2 and to describe the dynamic of the fund F. We opt for the standard model for the evolution of $F(t)$: we suppose that $F(t)$ follows a dynamic (under the physical measure P) given by

$$dF(t) = \mu \, F(t) \; dt + \sigma \, F(t) \; dW(t) \tag{5.1}$$

with $F(0) = 1$. The parameters μ and σ are the expectation and the standard deviation of the instantaneous yield rate of the fund respectively, and $\{W(t)\}_{t \geq 0}$ is a standard Brownian motion. In this "Gaussian world," everything is well known. The equivalent martingale measure Q_2 is given by

$$Q_2(D_t) = E_{P_2}\left(e^{-\int_0^t \left(\frac{\mu - r(s)}{\sigma}\right) dW(s) - 0.5 \int_0^t \left(\frac{\mu - r(s)}{\sigma}\right)^2 ds} I_{D_t} \right) \tag{5.2}$$

for any event $D_t \in \Im_t^2$.

Under the measure Q_2 and a (new) Brownian motion $\widetilde{W}(t) = W(t) - \int_0^t \left(\frac{\mu - r(s)}{\sigma}\right) ds$, the dynamic of F is given by

$$dF(t) = r(t)\ F(t)\ dt + \sigma F\ (t)\ d\widetilde{W}(t)\ . \tag{5.3}$$

For $l = 1, \ldots, T$, $G \in \mathbb{R}$, we have

$$\begin{aligned} &E_{Q_2}\left(P(0,l)(G - S(l))_+\right) \\ &\quad = P(0,l)\ E\left(G - \sum_{u=0}^{l-1} A_u^{(l)} e^{-0.5\,\sigma^2(l-u)+\sigma\left(\tilde{W}(l)-\tilde{W}(u)\right)}\right)_+ , \end{aligned} \tag{5.4}$$

where the expectation is the one induced by the Brownian motion $\widetilde{W}(t)$, $t \geq 0$. The factors inside the sum in (5.4) can be expressed as

$$\begin{aligned} A_0^{(l)} &= (S(0) + P_{sa}(0))\ e^{-\delta\, l} \frac{1}{P(0,l)} , \\ A_u^{(l)} &= P_{sa}(u)\ e^{-\delta(l-u)} \frac{P(0,u)}{P(0,l)} , \qquad u \geq 1 . \end{aligned}$$

It should be noted that $\sum\limits_{u=0}^{l-1} A_u^{(l)} = E_{Q_2}(S(l))$. If we define

$$a_u^{(l)} = \frac{A_u^{(l)}}{E_{Q_2}\ (S(l))} , \quad g_D^{(l)} = \frac{G_D}{E_{Q_2}\ (S(l))} , \quad g_L^{(T)} = \frac{G_L}{E_{Q_2}\ (S(T))} \tag{5.5}$$

then the Asian put options that have to be valued have the following forms ($l = 1, \ldots, T; * = L$ or D)

$$P(0,l)\ E_{Q_2}\left((G_* - S(l))_+\right) = P(0,l)\ E_{Q_2}(S(l))\ E\left(g_*^{(l)} - s_l\right)_+ \tag{5.6}$$

with

$$s_l = \sum_{u=0}^{l-1} a_u^{(l)}\ e^{-0.5\sigma^2(l-u)+\sigma\, X_u^{(l)}} \tag{5.7}$$

for a Gaussian random vector $X^{(l)} \sim N(0, C^{(l)})$ with covariance matrix given by

$$C^{(l)} = \begin{bmatrix} l & l-1 & l-2 & \cdots & 1 \\ l-1 & l-1 & l-2 & & \vdots \\ l-2 & l-2 & l-2 & \cdots & 1 \\ \vdots & & & & \vdots \\ 1 & & \cdots & & 1 \end{bmatrix}. \tag{5.8}$$

Hence, the problem of valuing these options consists in determining the distribution of a weighted average, with weights $a^{(l)} = (a_u^{(l)}, u = 0, \ldots, l-1)$, of correlated log-normal random variables with a common mean of 1. In what follows, we denote by $C_{i,j}^{(l)}$, $i, j = 0, \ldots, l-1$, the element $(i+1)$, $(j+1)$ of the matrix $C^{(l)}$.

Remark With a flat interest rate structure and saving premiums $P_{sa}(u) = 0$, the above formula is nothing else than Black and Scholes formula with a dividend rate δ.

6 Calculation methods

The main difficulty in determining a market consistent value for the contract is therefore the valuation of a series of Asian put options that define the guarantees at maturity or death while the contract is in force. Even in the simple model that we chose, there are no closed analytical formulas for the value of that kind of options.

Then, to compute the value of the option we have to resort to numerical simulation of the expectation using, for example, Monte-Carlo simulations for the sample paths of the fund or by valuing the multiple integrals using sampling methods like lattice rules (see [4]). The problem with numerical simulations is the time required for the valuation of a portfolio with a lot of contracts, since an adequate stability level for the results is only achieved through a large number of simulations. Ideally, we should find an analytical upper bound, as close as possible to the true value, and of which we know a good bound for the error.

Unfortunately, there is no unique method that provides good approximations in all the cases we may envisage (different σ, long or short duration, high or low guarantee, ...). We present, in addition to the purely numerical approach, three simple methods. Two of our simple methods provide upper bounds. The third one is an approximation. The validity of these three methods will be discussed through a numerical example in the following section.

1. Numerical simulation

There are different ways to perform the numerical simulations (see [4]). The most usual of them consists in simulating different paths for s_l under the risk-neutral measure Q_2.
We simulate s_l in a recursive way using the relation (see (5.7))

$$s_l(t) = e^{-0.5\,\sigma^2 + \sigma\, Z_t}\left(s_l(t-1) + a_{t-1}^{(l)}\right)$$

with $t = 1, \ldots, l$, $s_l(0) = 0$ and $s_l(l) = s_l$. The vector $Z = (Z_1, \ldots, Z_l)$ is a Gaussian vector $N(0, \mathbf{1})$. We select the Box-Muller method to generate Gaussian random variates and also utilize antithetic variables. Then, using Z_t^k, $k = 1, \ldots, N$ (number of simulations), $t = 1, \ldots, l$, a set of independent increments, we compute

$$s_l^{k,+}(t) = e^{-0.5\,\sigma^2 + \sigma Z_t^k}\left(s_l^{k,+}(t-1) + a_{t-1}^{(l)}\right)$$
$$s_l^{k,-}(t) = e^{-0.5\,\sigma^2 - \sigma Z_t^k}\left(s_l^{k,-}(t-1) + a_{t-1}^{(l)}\right).$$

For $l = 1, \ldots, T$, and $* = L$ or D, we have

$$\begin{aligned} &E_{Q_2}\left((G_* - S(l))_+\right) \\ &\quad = E_{Q_2}(S(l))\,(2N)^{-1}\sum_{k=1}^{N}\left(\left(g_*^{(l)} - s_l^{k,+}\right)_+ + \left(g_*^{(l)} - s_l^{k,-}\right)_+\right) \\ &\quad\quad + \varepsilon(N) \end{aligned}$$

with an error $\varepsilon(N) \to 0$ as $N \to \infty$.
The optimal number N of sample paths depends on three parameters g (level of the guarantee), l (time when the option is exercised) and σ (the instantaneous volatility of the fund).

2. Geometric approximation

The easiest way to obtain an upper bound is to use the convexity of the exponential function: an arithmetic mean is always greater than (or equal to) a geometric mean. Thus, we have almost surely (see also [1] for generalizations)

$$\sum_{u=0}^{l-1} a_u^{(l)} e^{-0.5\,\sigma^2(l-u)+\sigma X_u^{(l)}} \geq e^{-0.5\,\sigma^2 \sum\limits_{u=0}^{l-1} a_u^{(l)}(l-u)+\sigma \sum\limits_{u=0}^{l-1} a_u^{(l)} X_u^{(l)}}$$
$$= e^{-0.5\,\Gamma_1^2+\Gamma_2 Z}$$

with $Z \sim N(0,1)$, $\Gamma_1^2 = \sigma^2 \left(l - \sum\limits_{u=0}^{l-1} u\, a_u^{(l)}\right)$ and $\Gamma_2 = \sigma \sqrt{\sum\limits_{u_1,u_2=0}^{l-1} a_{u_1}^{(l)} a_{u_2}^{(l)} C_{u_1,u_2}^{(l)}}$.

Consequently, with ($l = 1,\ldots,T$; $* = L$ or D) and, as usual, $\phi(x) = (2\pi)^{-1/2} \int\limits_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$, we have the "universal" bound

$$\begin{aligned} &E_{Q_2}\left((G_* - S(l))_+\right) \\ &\quad\leq E_{Q_2}(S(l))\, E\left(g_*^{(l)} - e^{-0.5\,\Gamma_1^2+\Gamma_2 Z}\right)_+ \\ &\quad= G_* \phi\left(\frac{\ln g_*^{(l)} + 0.5\,\Gamma_1^2}{\Gamma_2}\right) \\ &\qquad - E_{Q_2}(S(l))\, e^{-0.5\left(\Gamma_1^2-\Gamma_2^2\right)}\, \phi\left(\frac{\ln g_*^{(l)} + 0.5\,\Gamma_1^2}{\Gamma_2} - \Gamma_2\right). \end{aligned} \qquad \text{(UP1)}$$

3. Edgeworth's approximation

In general, a method for approximating (unknown) distributions is the Edgeworth's expansion. One possibility (originally developed by Jarrow & Rudd in a general setup) is then to approximate the distribution of s_l with a log-normal distribution (see, for example, references in [5]). With this method one compares the series expansion of the logarithm of the Fourier transform of the (unknown) probability distribution of s_l with the series expansion of the logarithm of the Fourier transform of a know probability distribution (a log-normal for example).

The first four terms in Edgeworth's expansion of the (unknown) density function $f_{s_l}(x)$ of s_l "around" a (known) density function $f_Y(x)$ with the same mean as

s_l are given by

$$f_{s_l}(x) \approx f_Y(x) + \frac{k_2(f_{s_l}) - k_2(f_Y)}{2} f_Y^{(2)}(x) - \frac{k_3(f_{s_l}) - k_3(f_Y)}{3!} f_Y^{(3)}(x) + \frac{k_4(f_{s_l}) - k_4(f_Y) + 3(k_2(f_{s_l}) - k_2(f_Y))^2}{4!} f_Y^{(4)}(x)$$

with $k_i(f)$ the i-th cumulant of the probability density function f and $f^{(i)}$ the i-th derivative of f.
The first four cumulants are given in term of the i-moment $m_i(f)$ by

$$\begin{aligned}
k_1(f) &= m_1(f) \\
k_2(f) &= m_2(f) - m_1(f)^2 \\
k_3(f) &= 2m_1(f)^3 - 3m_1(f)\,m_2(f) + m_3(f) \\
k_4(f) &= -6m_1(f)^4 + 12m_1(f)\,m_2(f) - 3m_2(f)^2 - 4m_1(f)\,m_3(f) \\
&\quad + m_4(f)\,.
\end{aligned}$$

We set also $Y_l = e^{-0.5\,B_l^2 + B_l Z}$, $Z \sim N(0,1)$, with B_l such that

$$E(Y_l^2) = e^{B_l^2} = E(s_l^2)\,.$$

With this assumption we have that

$$\begin{aligned}
k_2(f_{s_l}) &= k_2(f_{Y_l}) \\
k_3(f_{s_l}) - k_3(f_{Y_l}) &= E(s_l^3) - E(Y_l^3) \\
k_4(f_{s_l}) - k_4(f_{Y_l}) &= E(s_l^4) - E(Y_l^4) - 4(E(s_l^3) - E(Y_l^3))\,.
\end{aligned}$$

We see that, for all $k > 0$, the k-moment of $s(l)$ is given by

$$\begin{aligned}
E(s_l^k) &= \sum_{u_1,\ldots,u_k=0}^{l-1} a_{u_1}^{(l)} \cdots a_{u_k}^{(l)} e^{-0.5\,\sigma^2((l-u_1)+\cdots+(l-u_k))} E\left(e^{\sigma\left(X_{u_1}^{(l)}+\cdots+X_{u_k}^{(l)}\right)}\right) \\
&= \sum_{u_1,\ldots,u_k=0}^{l-1} a_{u_1}^{(l)} \cdots a_{u_k}^{(l)}\, e^{-0.5\,\sigma^2((l-u_1)+\cdots+(l-u_k))}\, e^{0.5\,\sigma^2\,\mathrm{Var}\left(X_{u_1}^{(l)}+\cdots+X_{u_k}^{(l)}\right)} \\
&= \sum_{u_1,\ldots,u_k=0}^{l-1} a_{u_1}^{(l)} \cdots a_{u_k}^{(l)} e^{0.5\,\sigma^2 \sum\limits_{\substack{i,j=1 \\ i\neq j}}^{k} C_{u_i,u_j}^{(l)}}
\end{aligned} \tag{6.1}$$

because

$$\mathrm{Var}\left(X_{u_1}^{(l)} + \cdots + X_{u_k}^{(l)}\right) - \sum_{i=1}^{k} (l - u_i)$$

$$= \sum_{i,j=1}^{k} E\left(X_{u_i}^{(l)} X_{u_j}^{(l)}\right) - \sum_{i=1}^{k} (l - u_i)$$

$$= \sum_{i,j=1}^{k} C_{u_i,u_j}^{(l)} - \sum_{i=1}^{k} C_{u_i,u_i}^{(l)} .$$

In particular

$$E\left(s_l^2\right) = E\left(\sum_{u=0}^{l-1} a_u^{(l)} e^{-0.5\,\sigma^2 (l-u) + \sigma X_u^{(l)}}\right)^2 = \sum_{u_1,u_2=0}^{l-1} a_{u_1}^{(l)} a_{u_2}^{(l)} e^{\sigma^2 (l-(u_1 \vee u_2))}$$

i.e.

$$B_l = \sqrt{\ln\left(\sum_{u_1,u_2=0}^{l-1} a_{u_1}^{(l)} a_{u_2}^{(l)} e^{\sigma^2 (l-(u_1 \vee u_2))}\right)} .$$

The k-th moment of Y_l is simply given by

$$E\left(Y_l^k\right) = e^{0.5\, B_l^2\, k(k-1)} .$$

When we only take the first term of the Edgeworth's expansion, we obtain the following approximation:

$$\begin{aligned} E_{Q_2}\left((G_* - S(l))_+\right) &\approx E_{Q_2}(S(l))\, E\left(g_*^{(l)} - Y_l\right)_+ \\ &= G_* \phi\left(\frac{\ln g_*^{(l)} + 0.5 B_l^2}{B_l}\right) \\ &\quad - E_{Q_2}(S(l))\, \phi\left(\frac{\ln g_*^{(l)} - 0.5 B_l^2}{B_l}\right) . \end{aligned} \tag{AP1}$$

Then, by adding the second and third non-zero terms in Edgeworth's expansion and using the fact that, for $i > 1$,

$$\int_{-\infty}^{g_*^{(l)}} \left(g_*^{(l)} - x\right) f_{Y_l}^{(i)}(x)\, dx = f_{Y_l}^{(i-2)}\left(g_*^{(l)}\right)$$

the following approximations respectively follow

$$E_{Q_2}\left((G_* - S(l))_+\right) \approx G_*\phi\left(\frac{\ln g_*^{(l)} + 0.5B_l^2}{B_l}\right) - E_{Q_2}(S(l))\,\phi\left(\frac{\ln g_*^{(l)} - 0.5B_l^2}{B_l}\right) + \Delta_1^l\left(g_*^{(l)},\sigma\right) \tag{AP2}$$

and

$$E_{Q_2}\left((G_* - S(l))_+\right) \approx G_*\phi\left(\frac{\ln g_*^{(l)} + 0.5B_l^2}{B_l}\right) - E_{Q_2}(S(l))\,\phi\left(\frac{\ln g_*^{(l)} - 0.5B_l^2}{B_l}\right) + \Delta_1^l\left(g_*^{(l)},\sigma\right) + \Delta_2^l\left(g_*^{(l)},\sigma\right). \tag{AP3}$$

The "corrections" Δ_1^l and Δ_2^l are given by

$$\Delta_1^l\left(g_*^{(l)},\sigma\right) = -\frac{E_{Q_2}(S(l))\left(E\left(s_l^3\right) - e^{3B_l^2}\right) f_{Y_l}^{(1)}\left(g_*^{(l)}\right)}{3!}$$

and

$$\Delta_2^l\left(g_*^{(l)},\sigma\right) = \frac{E_{Q_2}(S(l))\left(\left(E\left(s_l^4\right) - e^{6B_l^2}\right) - 4\left(E\left(s_l^3\right) - e^{3B_l^2}\right)\right) f_{Y_l}^{(2)}\left(g_*^{(l)}\right)}{4!}$$

where the third and the fourth moments of $s(l)$ are given by (6.1) and $f_{Y_l}^{(i)}$, $i = 1, 2$, are the first and second derivatives of the probability density function of Y_l given by

$$f_{Y_l}(x) = \frac{1}{x\sqrt{2\pi B_l^2}}\, e^{-\frac{\ln(x)^2}{2B_l^2}}, \qquad x > 0.$$

4. Approximation by a weighted average of European put options

In this approach, we approximate the Asian put option by a weighted average of European put options. We assume that all the weights $a_u^{(l)}$ are positive.

With $\Lambda = \{\lambda \in \mathbb{R}^l \mid \lambda_i \geq 0; \sum_{i=0}^{l-1} \lambda_i = 1\}$, we have

$$E\left(g_*^{(l)} - \sum_{u=0}^{l-1} a_u^{(l)} e^{-0.5\,\sigma^2(l-u)+\sigma\, X_u^{(l)}}\right)_+ \leq \inf_{\lambda\in\Lambda} \sum_{u=0}^{l-1} a_u^{(l)} E\left(\frac{\lambda_u}{a_u^{(l)}}\, g_*^{(l)} - e^{-0.5\,\sigma^2(l-u)+\sigma\, X_u^{(l)}}\right)_+ .$$

Thus, the idea consists in determining weights λ_u^0 such that the minimum of the expression on the right hand side is reached. Let's denote by $f^{(u)}(k)$ the value of the put option

$$f^{(u)}\,(k) = E\left(k - e^{-0.5\,\sigma^2(l-u)+\sigma\, X_u^{(l)}}\right)_+ .$$

We define the Lagrangian

$$L\,(\lambda,\varphi) = \sum_{u=0}^{l-1} a_u^{(l)} f^{(u)} \left(\frac{\lambda_u}{a_u^{(l)}}\, g_*^{(l)}\right) - \varphi\left(\sum_{u=o}^{l-1} \lambda_u - 1\right).$$

Using

$$\frac{d}{dk} f^{(u)}\,(k) = P\left(e^{-0.5\,\sigma^2(l-u)+\sigma\, X_u^{(l)}} \leq k\right) = \phi\left(\frac{\ln k + 0.5\,\sigma^2\,(l-u)}{\sigma\,\sqrt{l-u}}\right)$$

a simple calculation shows that, for all $0 < \varphi < g_*$, L is extremal if

$$\lambda_u = \lambda_u\,(\varphi) = \frac{a_u^{(l)}}{g_*^{(l)}}\, e^{-0.5\,\sigma^2(l-u)+\sigma\,\sqrt{(l-u)}\,\phi^{-1}\left(\frac{\varphi}{g_*^{(l)}}\right)} .$$

Also, if $\varphi = \varphi^0$ such that $\sum \lambda_u(\varphi^0) = 1$, we obtain

$$\sum_{u=0}^{l-1} a_u^{(l)} E\left(\frac{\lambda_u\,(\varphi^0)}{a_u^{(l)}} g_*^{(l)} - e^{-0.5\,\sigma^2(l-u)+\sigma\, X_u^{(l)}}\right)_+ = \varphi^0 - \sum_{u=0}^{l-1} a_u^{(l)}\, \phi\left(\phi^{-1}\left(\frac{\varphi^0}{g_*^{(l)}}\right) - \sigma\sqrt{l-u}\right).$$

The upper bound is also given by

$$E_{Q_2}\left(\left(G_* - S(l)\right)_+\right)$$
$$\leq E_{Q_2}(S(l))\,\varphi^0 - E_{Q_2}(S(l))\sum_{u=0}^{l-1} a_u^{(l)}\phi\left(\phi^{-1}\left(\frac{\varphi^0}{g_*^{(l)}}\right) - \sigma\sqrt{l-u}\right). \tag{UP2}$$

The challenge is also to estimate $0 < \varphi^0 < g_*$ such that

$$\sum_{u=0}^{l-1} a_u^{(l)}\, e^{-0.5\,\sigma^2(l-u)+\sigma\,\sqrt{(l-u)}\,\phi^{-1}\left(\frac{\varphi^0}{g_*^{(l)}}\right)} = g_*^{(l)}$$

which is numerically not so difficult.

7 Numerical example

As an example, we consider an «endowment»-like contract ($G_D = G_L = G$) with a guaranteed interest rate i.
The premium is given by (in standard actuarial notation)

$$P_{tot} = G \cdot \frac{A_{x:n} + \alpha + \gamma \cdot \ddot{a}_{x:n}}{\ddot{a}_{x:n}},$$

implying that, for simplicity, we disregard "β-costs".
The saving premium is simply given by

$$P_{sa}(l) = G\left(v\left(1 - \frac{\ddot{a}_{x+l+1:n-(l+1)}}{\ddot{a}_{x:n}}\right) - \left(1 - \frac{\ddot{a}_{x+l:n-l}}{\ddot{a}_{x:n}}\right)\right)$$

with $v = (1+i)^{(-1)}$.
The deduction in case the cash value option is exercised is given by the unamortized underwriting expenses (Zillmer deduction), that is

$$EZ(l) = 0.04 \cdot G\frac{\ddot{a}_{x+l:n-l}}{\ddot{a}_{x:n}}.$$

For our example, we consider an "average" male insured who bought a policy 5 years ago at the age $x = 35$ years for a duration of $n = 30$ years and a face amount of $G = 100'000$ monetary units.

As "fixed" parameters for our analyses, we select:

Mortality table for pricing	EKM 95
Best estimate assumption for mortality	60% EKM 95
α costs	4%
γ costs	0.5%
Best estimate costs $((K(t))$	300
Management fee (δ)	1%
Average annual yield rate of the fund during the last 5 years	7%

We take $i = 2\%$ as basis value for the guaranteed interest rate.
To compare the accuracy of the approximations presented above we will also consider the "extremes cases" $i = 0\%$ (low guarantee) and $i = 5\%$ (high guarantee).

Given these parameters, we have

	Basis value $(i = 2\%)$	Low guarantee $(i = 0\%)$	High guarantee $(i = 5\%)$
Total premium (P_{tot})	3'271	4'127	2'378
Value of the assets in the fund at the valuation date $(S(0))$	15'668	21'405	9'503

Recall that the future evolution of one unit of the fund with respect to the "risk neutral measure" is a function of the yield curve and the volatility (see (5.3)).

As variable parameters, we use also:

	Basis values	Sensitivities
Zero-coupon bond at time 0	given by yield curve at 1.1.2006 (Swiss Government bonds)	Parallel move of $\xi \in [-1\%, 5.5\%]$
Lapse rate	$h_t = h = 4\%$ (constant over the time)	Relative variation $\eta \in [-50\%, 75\%]$
Volatility	$\sigma = 10\%$	$\sigma \in [0, 35\%]$

In what follows, we consider only the two components $MV_1(0)$ (the value of the guarantee at maturity) and $MV_2(0)$ (the value of the "risk plus cost plus surrender" process).
We denote by $-RBC(0)$ the sum of these two components, by analogy to the Risk Bearing Capital of the Swiss Solvency Test (see remark 3. of section 4.);

the market consistent value at the valuation date for the contract is simply given by $S(0) - RBC(0)$ (see (4.1)).

1. Numerical simulations

For the basis values of the parameters, we find (with 100'000 simulations) the following values:

Value of the guarantee at maturity ($MV_1(0)$)	$2'591$
Value of the „risk plus cost plus surrender" process ($MV_2(0)$)	$-6'462$
Total ($-RBC(0)$)	$-3'871$

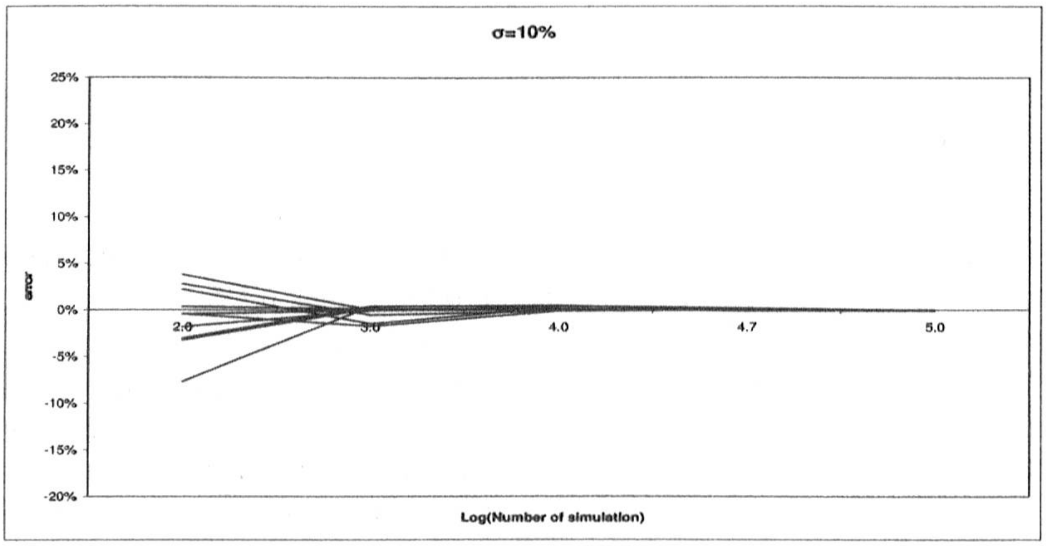
As a percentage of the insured capital, the value of the guarantee at maturity costs 2.6%, the benefits and expenses process brings 6.5%.
Thus, the sum $MV_1(0) + MV_2(0)$ corresponds to -3.87% of the insured capital. The larger weight of $MV_2(0)$ is due to the margins on the expense charges: on the one hand, the underwriting costs (α-costs) are amortized, and on the other hand we have supposed the inclusion of a margin of 0.2% of the insured capital into the annual "expenses" (γ-costs). It is interesting to compare this value, -3.87%, obtained by stochastic simulation to the various approximations we suggested. The next table presents the corresponding values, again expressed as a percentage of the insured capital:

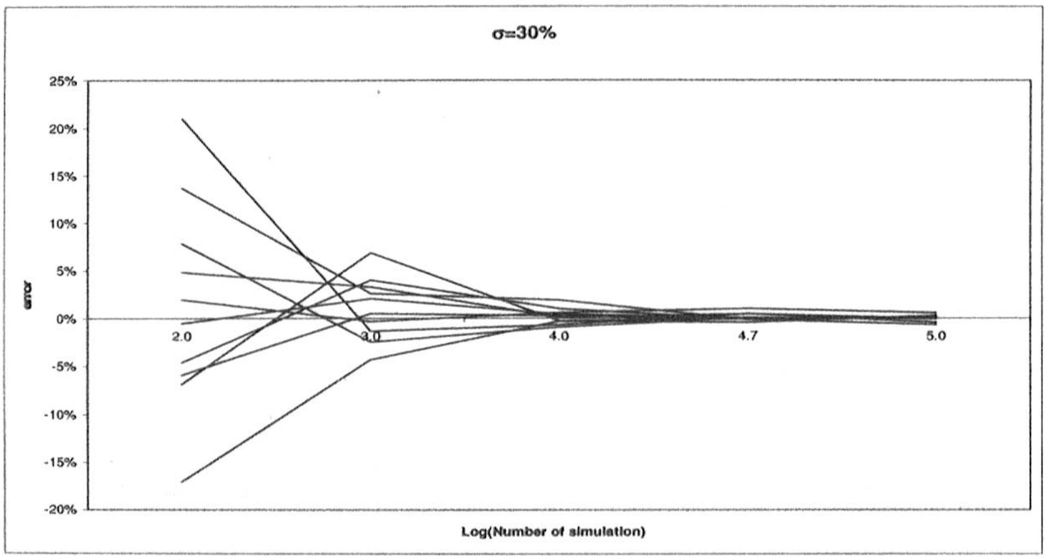
	$-RBC(0)$ (in % of the insured capital)
Approximation by a weighted average of European put options (UP2)	-3.58%
Geometric approximation (UP1)	-3.68%
Edgeworth's approximation (AP2)	-3.79%
Edgeworth's approximation (AP1)	-3.83%
Numerical simulations	**-3.87%**
Edgeworth's approximation (AP3)	-4.69%

We see that the best approach is the one consisting in approximating the value of the assets in the fund by a log-normal random variable with the same first two moments (Edgeworth's approximation (AP1))

For these analyses, we simulated 100′000 future evolutions of the fund.

In the next two graphs, we show the relative error of 10 different "random experiments" as a function of the number of simulations for the basis values ($\sigma = 10\%$) and an extreme case ($\sigma = 30\%$). The errors of these 10 different "random experiments" are calculated with respect to the mean value of others 10 different "random experiments" with each 10^5 simulations of the future evolution of the fund. The errors are expressed as a function of the Log of the number of simulations N with $N = 100, 1000, 10'000, 50'000$ and $100'000$.

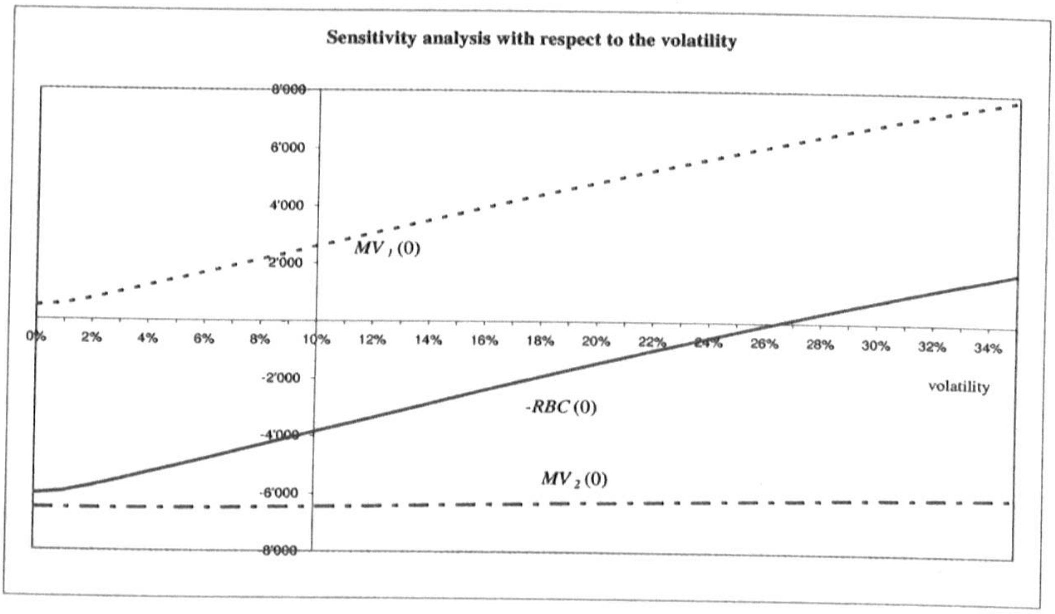




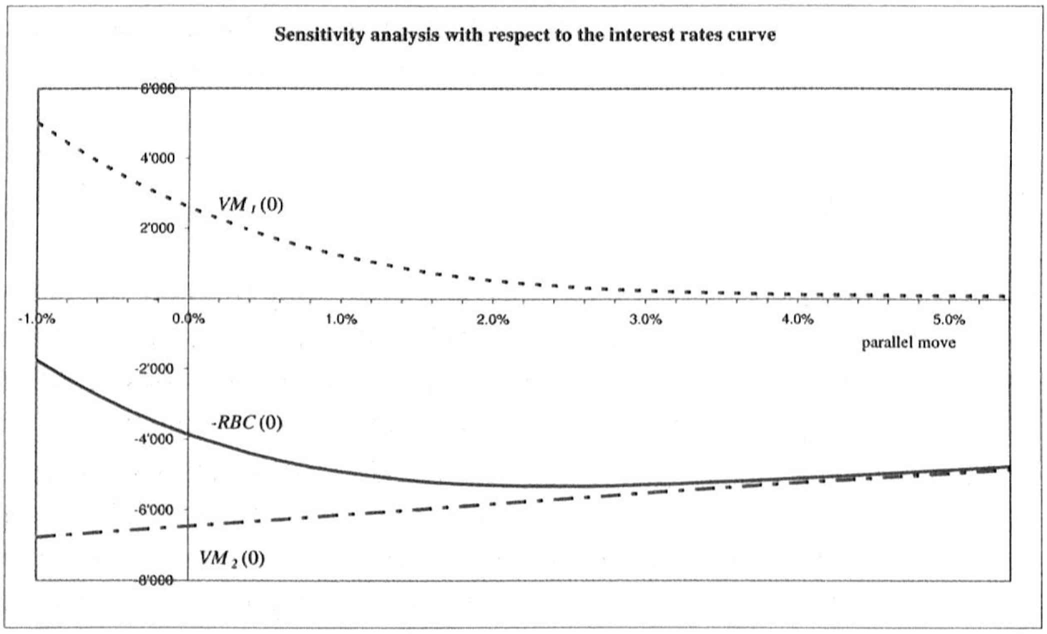
For the extreme case, $50'000$ simulations are needed to obtain a 1% accuracy (i.e. a relative error less than 1%); with only $10'000$ simulations, we have a 1.5% accuracy, while a $1'000$ sample path simulation can give an "accuracy" as high as 7%. For the basis value case, the simulation is already stable with $10'000$ simulations, while $1'000$ simulations produce 2% relative errors.

2. *Sensitivity analysis*

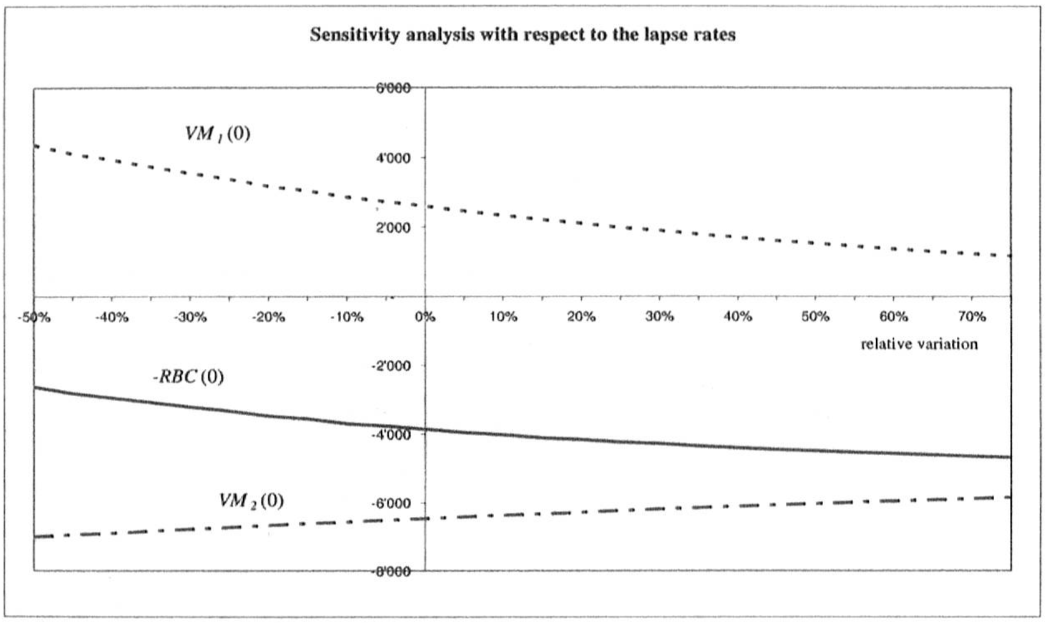
From a risk management point of view, it is interesting to know the behaviour of the value of the guarantee at maturity ($MV_1(0)$) and the value of the "risk plus cost plus surrender" process ($MV_2(0)$) with respect to variations of different risk factors. In the figures that follow we have taken $10'000$ simulations.
In regard to the volatility, $-RBC(0) = MV_1(0) + MV_2(0)$ reacts almost linearly. We also observe a relative stability of $MV_2(0)$: the difference between the minimal value of $MV_2(0)$ (when $\sigma = 0\%$) and the extreme value of $MV_2(0)$ (when $\sigma = 35\%$) is about 0.5% of the insured capital. For $MV_1(0)$, the corresponding difference amounts to 7.5% of the insured capital. This shows that, in practice, it is justified to place the modeling effort on the guarantee for the maturity benefit while using a deterministic model for the death benefit (i.e. with $\sigma = 0\%$).



The next figure shows the sensitivity of $MV_1(0)$, $MV_2(0)$, and $-RBC(0)$ with respect to a parallel move of magnitude $\xi \in [-1\%, 5.5\%]$ of the interest rates curve. One should notice the asymmetric behaviour of $-RBC(0)$ with the risk free rate.



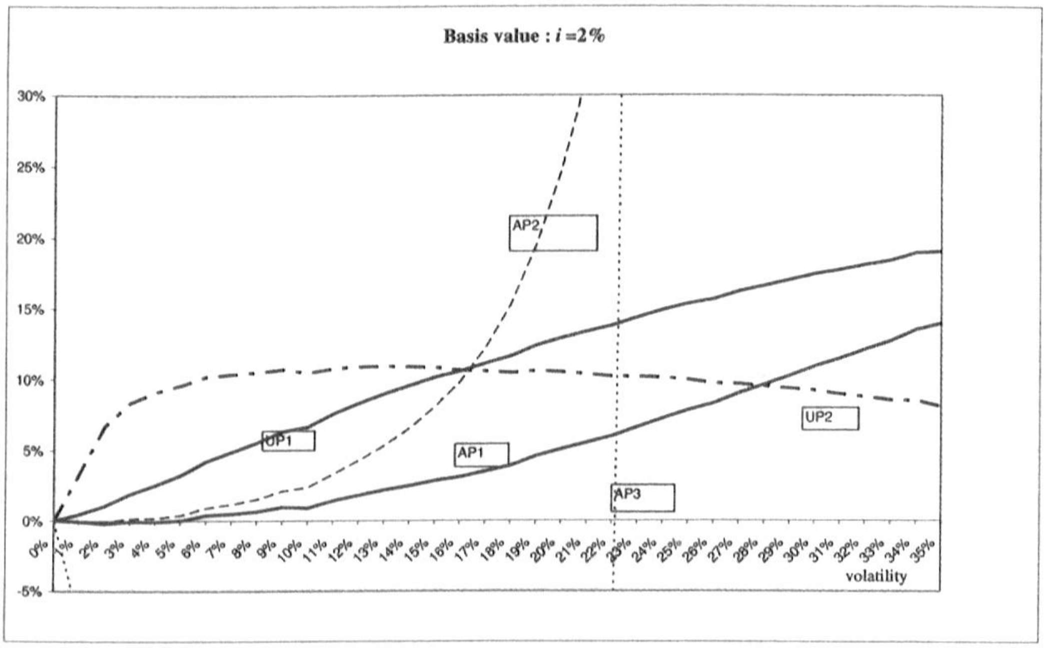
However, the behaviour of $-RBC(0)$ with respect to withdrawal rate variations is almost linear. The next graph shows the variation of the central values resulting from a relative variation $\eta \in [-50\%, 75\%]$ of the withdrawal rates.



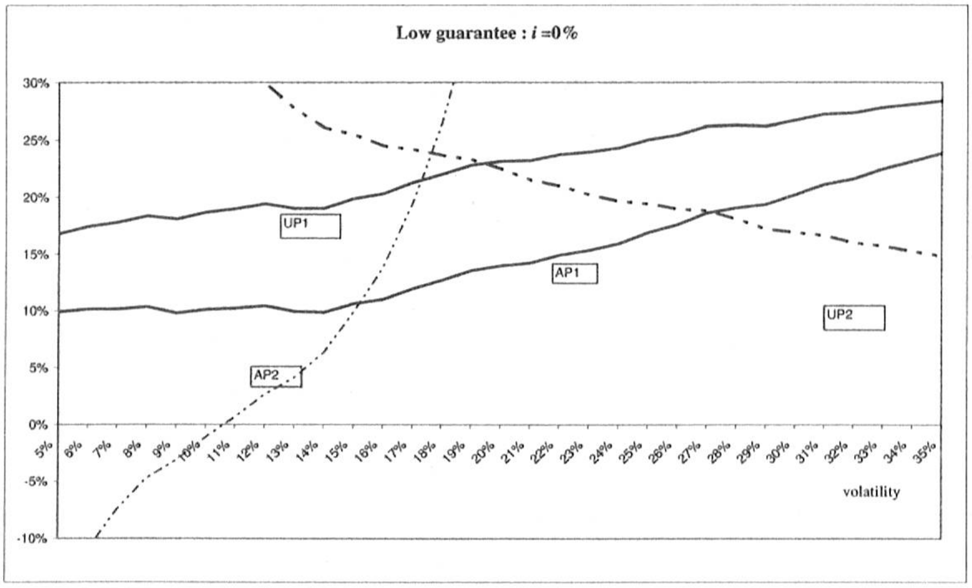
3. Comparisons of approximations

We examine the accuracy of the approximations presented above.

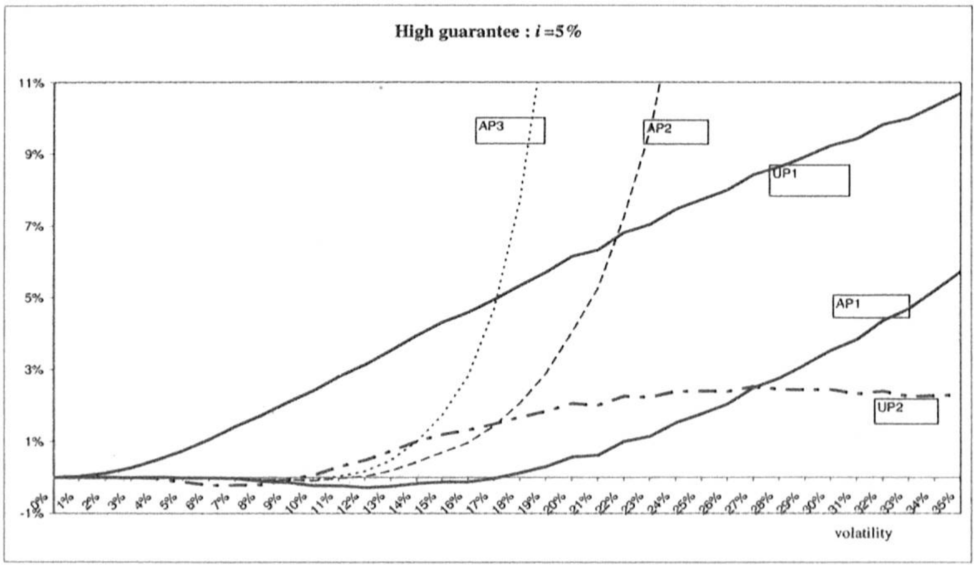
We are confining ourselves to the case of a maturity benefit-only contract, i.e. $MV_1(0)$, and we analyze the approximation errors as functions of the volatility level for three situations: $i = 0\%, 2\%$ and 5%. These situations correspond to the three possibilities: $g^{(T)}{}_L < E(s_T)$, $g^{(T)}{}_L \approx E(s_T)$, and $g^{(T)}{}_L > E(s_T)$ (see (5.5), (5.6) and (5.7)).
In the figures that follow, we have for each approximation the relative error computed with respect to the Monte-Carlo simulation results (with $N = 50'000$). The fact that the graphs are not absolutely smooth comes from the way the Monte-Carlo results have been obtained: for each level of σ a new sample of random variates for the fund has been generated. However, the accuracy is sufficient to compare the different approximation methods.
For the basis values ($i = 2\%$), we observe that the geometric bound, (UP1), is not good. Up to values around 28%, the best method is the one where the sum of log-normal random variables is approximated by a log-normal random variable with the same first two moments (approximation (AP1)). The relative error is less that 3% for low volatility levels (levels less than 15%). For high volatilities, the best approximation is given by the bound (UP2), which is valuing the Asian put option by a weighted average of European put options.
The Edgeworth's expansion with higher order correction terms is very unstable and does not provide more accuracy.



For the case $i = 0\%$ we obtain the following relative errors presented in graphical form:



For low level volatilities, the best approximation is (AP2). In this case, the first correction term in Edgeworth's expansion provides the hoped-for accuracy. We do not show the results from the Edgeworth's expansion with the second correction term (AP3), because they strongly diverge. For high volatilities, approximation (UP2) is again the best. One should notice that all these approximations are inaccurate. The reason is the following: if ${g^{(T)}}_L < E(s_T)$, the distribution's tail of s_T becomes more and more important in the valuation.
For the case $i = 5\%$, we have the following errors presented in the following figure:



When the volatilities are less than 15%, all the methods can be considered good, except the so-called universal geometric bound. For high volatilities, the (UP2) method remains the best. One can observe that we can choose approximation methods that keep the error within a 3% range for all volatility values.

8 Conclusion

This simple example shows why it is reasonable to decompose the "policy's market consistent value" of a guaranteed unit linked insurance policy with periodic premiums as the sum of three components (the market value of the fund, the value of the guarantee at maturity, the value of the "risk plus cost plus surrender" process) and why in practice, it is justified to calculate only the value of the guarantee at maturity while using a deterministic model for the death benefit.
We show also that it is not so obvious to devise a simple general method to find an analytical approximation for the value of the guarantee at maturity that provides good approximations in the cases of different volatilities and high or low guarantees.
For small volatilities, the best approach is the one consisting in approximating the value of the fund by a log-normal random variable with the same first two moments; one step further might consist in using the Edgeworth's series. Unfortunately, it happens that this is useless: the high order terms are too unstable. On the other hand, for high volatilities, the best approach is to bound the Asian put option by a weighted average of European put options.

Appendix

Proof of (4.1)

With $Q = P \times Q_2$ and $k(l) = A(l)$ or $D(l)$ or $O(l)$, $l \geq t$, we have

$$E_Q\left(I_{k(l)}\beta(l)^{-1} S(l) \mid \Im_t\right) = E_P\left(I_{k(l)} \mid \Im_t^1\right) E_{Q_2}\left(\beta(l)^{-1} S(l) \mid \Im_t^2\right)$$

and

$$\begin{aligned}
E_P\left(I_{A(l)} \mid \Im_t^1\right) &= I_{A(t)} M(t,l)_{1,1} = I_{A(t)}\, {}_t p_{l-1}\, m(l-1)_{1,1} \ , \\
E_P\left(I_{O(l)} \mid \Im_t^1\right) &= I_{A(t)}\, {}_t p_{l-1}\, m(l-1)_{1,2} \ , \\
E_P\left(I_{D(l)} \mid \Im_t^1\right) &= I_{A(t)}\, {}_t p_{l-1}\, m(l-1)_{1,3} \ .
\end{aligned}$$

We first consider the outgoing cash flows, with $CF^{O1}(t) = I_{\{t>0\}} I_{O(t)} S(t)$

$$\begin{aligned}
&\sum_{l=t}^{T} E_Q\left(\beta(l)^{-1}\left(CF^L(l) + CF^{O1}(l) + CF^D(l) + CF^C(l)\right) \mid \Im_t\right) \\
&= I_{A(t)}\, {}_t p_T E_{Q_2}\left(\beta(T)^{-1} S(T) \mid \Im_t^2\right) \\
&\quad + I_{A(t)} \sum_{l=t+1}^{T} \left({}_t p_{l-1}\left(m(l-1)_{1,2} + m(l-1)_{1,3} + e^{\delta} - 1\right)\right) \\
&\qquad \cdot E_{Q_2}\left(\beta(l)^{-1} S(l) \mid \Im_t^2\right) \\
&\quad + I_{A(t)} \sum_{l=t+1}^{T} {}_t p_{l-1} K(l) E_{Q_2}\left(\beta(l)^{-1} \mid \Im_t^2\right) \\
&\quad + I_{A(t)}\, {}_t p_T E_{Q_2}\left(\beta(T)^{-1}\left(G_L - S(T)\right)_+ \mid \Im_t^2\right) \\
&\quad + I_{A(t)} \sum_{l=t+1}^{T} {}_t p_{l-1} m(l-1)_{1,3} E_{Q_2}\left(\beta(l)^{-1}\left(G_D - S(l)\right)_+ \mid \Im_t^2\right) . \qquad \text{(A1)}
\end{aligned}$$

By construction, for $l > t$, we have

$$S(l) = S(t)\, e^{-\delta(l-t)} \frac{F(l)}{F(t)} + \sum_{u=t}^{l-1} P_{sa}(u)\, e^{-\delta(l-u)} \frac{F(l)}{F(u)} \ .$$

Since $\{\beta(s)^{-1} F(s)\}_{s\geq 0}$ is a $(\Im^2, Q_2)$ martingale, it follows that

$$E_{Q_2}\left(\beta(l)^{-1}\frac{F(l)}{F(u)}\,\middle|\,\Im_t^2\right) = E_{Q_2}\left(E_{Q_2}\left(\beta(l)^{-1}\frac{F(l)}{F(u)}\,\middle|\,\Im_u^2\right)\,\middle|\,\Im_t^2\right)$$
$$= E_{Q_2}\left(\beta(u)^{-1}\,\middle|\,\Im_t^2\right).$$

Consequently, for all $l > t$,

$$E_{Q_2}\left(\beta(l)^{-1} S(l)\,\middle|\,\Im_t^2\right) = \beta(t)^{-1} S(t)\, e^{-\delta(l-t)}$$
$$+ \sum_{u=t}^{l-1} P_{sa}(u)\, e^{-\delta(l-u)} E_{Q_2}\left(\beta(u)^{-1}\,\middle|\,\Im_t^2\right).$$

Given the stochastic nature of the m matrix, the first two lines of (A1) lead to

$${}_tp_T E_{Q_2}\left(\beta(T)^{-1} S(T)\,|\,\Im_t^2\right)$$
$$+ \sum_{l=t+1}^{T} \left({}_tp_{l-1}\left(e^{\delta} - m(l-1)_{1,1}\right)\right) E_{Q_2}(\beta(l)^{-1} S(l)\,|\,\Im_t^2)$$
$$= \beta(t)^{-1} S(t)\, {}_tp_T\, e^{-\delta(T-t)} + \beta(t)^{-1} S(t) \sum_{l=t+1}^{T} \left({}_tp_{l-1} e^{\delta} - {}_tp_l\right) e^{-\delta(l-t)}$$
$$+ {}_tp_T \sum_{u=t}^{T-1} P_{sa}(u)\, e^{-\delta(T-u)} E_{Q_2}(\beta(u)^{-1}\,|\,\Im_t^2)$$
$$+ \sum_{l=t+1}^{T} \sum_{u=t}^{l-1} {}_tp_{l-1}\left(e^{\delta} - m(l-1)_{1,1}\right) P_{sa}(u)\, e^{-\delta(l-u)} E_{Q_2}(\beta(u)^{-1}\,|\,\Im_t^2).$$

By using the following identity

$$\sum_{l=t+1}^{T} \sum_{u=t}^{l-1} {}_tp_{l-1}\left(e^{\delta} - m(l-1)_{1,1}\right) P_{sa}(u)\, e^{-\delta(l-u)} E_{Q_2}(\beta(u)^{-1}\,|\,\Im_t^2)$$
$$= \sum_{u=t}^{T-1} \left({}_tp_{T-1} e^{\delta} - {}_tp_T\right) P_{sa}(u)\, e^{-\delta(T-u)} E_{Q_2}\left(\beta(u)^{-1}\,|\,\Im_t^2\right)$$
$$+ \sum_{u=t}^{T-2} P_{sa}(u) E_{Q_2}(\beta(u)^{-1}\,|\,\Im_t^2) \sum_{l=u+1}^{T-1} \left({}_tp_{l-1} e^{\delta} - {}_tp_l\right) e^{-\delta(l-u)}$$

we obtain

$$
\begin{aligned}
& {}_tp_{T-1}\, m\,(T-1)_{1,1}\, E_{Q_2}\left(\beta\,(T)^{-1}\, S\,(T) \,\middle|\, \Im_t^2\right) \\
& \quad + \sum_{l=t+1}^{T} \left({}_tp_{l-1}\left(e^{\delta} - m\,(l-1)_{1,1}\right)\right) E_{Q_2}\left(\beta\,(l)^{-1}\, S\,(l) \,\middle|\, \Im_t^2\right) \\
& = \beta\,(t)^{-1}\, S\,(t) + {}_tp_{T-1}\, e^{\delta} \sum_{u=t}^{T-1} P_{sa}\,(u)\, e^{-\delta(T-u)} E_{Q_2}\left(\beta\,(u)^{-1} \,\middle|\, \Im_t^2\right) \\
& \quad + \sum_{u=t}^{T-2} {}_tp_u\, P_{sa}\,(u) E_{Q_2}\left(\beta\,(u)^{-1} \,\middle|\, \Im_t^2\right) \\
& \quad - {}_tp_{T-1}\, e^{\delta} \sum_{u=t}^{T-2} P_{sa}\,(u) E_{Q_2}\left(\beta\,(u)^{-1} \,\middle|\, \Im_t^2\right) e^{-\delta(T-u)} \\
& = \beta\,(t)^{-1}\, S\,(t) + \sum_{u=t}^{T-1} {}_tp_u\, P_{sa}\,(u) E_{Q_2}\left(\beta\,(u)^{-1} \,\middle|\, \Im_t^2\right) \quad .
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
& \sum_{l=t}^{T} E_Q\left(\beta\,(l)^{-1}\left(CF^L\,(l) + CF^{O1}\,(l) + CF^D\,(l) + CF^C\,(l)\right) \middle| \Im_t\right) \\
& \quad = I_{A(t)}\, \beta\,(t)^{-1}\, S\,(t) + \sum_{l=t}^{T-1} E_Q\left(\beta\,(l)^{-1}\, CF^{P_{sa}}\,(l) \,\middle|\, \Im_t\right) \\
& \qquad + I_{A(t)}\, {}_tp_T\, E_{Q_2}\left(\beta\,(T)^{-1}\left(G_L - S\,(T)\right)_+ \,\middle|\, \Im_t^2\right) \\
& \qquad + I_{A(t)} \sum_{l=t+1}^{T} {}_tp_{l-1}\, m\,(l-1)_{1,3} E_{Q_2}\left(\beta\,(l)^{-1}\left(G_D - S\,(l)\right)_+ \,\middle|\, \Im_t^2\right) \\
& \qquad + I_{A(t)} \sum_{l=t+1}^{T} {}_tp_{l-1}\, K\,(l) E_{Q_2}\left(\beta\,(l)^{-1} \,\middle|\, \Im_t^2\right) .
\end{aligned}
$$

To complete the proof, we only need to subtract from the last expression the value of future premiums

$$E_Q\left(\sum_{l=t}^{T-1}\beta(l)^{-1}\left(CF^{P_{sa}}(l)+CF^{P_r}(l)\right)\middle|\,\Im_t^2\right)$$

and the cash value that amounts to

$$E_Q\left(\sum_{l=t+1}^{T}\beta(l)^{-1}EZ(l)\,\middle|\,\Im_t^2\right).$$

Acknowledgements

I would like to thank François Dufresne for the precious support in the editing of the English version of this article.

Alessandro Jori
Groupe Mutuel
Rue du Nord 5
1920 Martigny

ajori@groupemutuel.ch

References

[1] Nielsen, J. and K. Sandmann (1996) "Uniqueness of the fair premium for equity-linked life insurance contracts," *The Geneva papers on risk and insurance theory* **21**.

[2] Bacinello, A.R. (2003) "Fair valuation of guaranteed life insurance participating contract embedding a surrender option," *The Journal of Risk and Insurance* **70** (3).

[3] Elliott, R. and P. Ekkehard Kopp (2000) *Mathematics of financial markets*, Springer, New York

[4] Boyle, P., Y Lai and K.S. Tan (2001) "Using lattice rules to value low-dimensional derivative contracts," AFIR 2001, Volume I, Toronto.

[5] Hellkvist, M. (2005) *Comparison of approximation methods for combinations of differently distributed random variables*, Helsinki University of Technology.

Abstract

With the introduction of the Swiss Solvency Test or the implementation of Solvency II, insurance companies have to attribute an economic value to their contracts. In this paper we consider the example of a unit linked insurance policy with periodic premiums and a guaranteed minimal amount in case of death and at maturity; for this type of contract, the main difficulty lies in the evaluation of these guarantees, which, from a financial point of view, corresponds to Asian type options. These options are not traded in the market and all the difficulty lies in the choice of an adequate model which allows to define and to calculate the value of these guarantees.
Even with a very simple market model, there are no closed analytical formulae and numerical simulations or analytical estimates are necessary. We present, in addition to the purely numerical approach, three different analytical methods. The validity of these three methods will be discussed through a numerical example.

Résumé

Avec l'introduction du Test Suisse de Solvabilité ou la mise en place de Solvency II, les compagnies d'assurances sont amenées à donner une valeur économique à leurs contrats d'assurance.
Dans cet article nous considérons l'exemple d'une assurance liée à des fonds de placement à primes périodiques avec des garanties en cas de vie et décès; pour ce type de contrats, la difficulté principale réside dans l'évaluation de ces garanties qui, du points de vue financier, correspondent à des options de type asiatique. Ces options ne sont pas échangées dans le marché et toute la difficulté réside dans le choix d'un modèle adéquat qui permet de définir et calculer la valeur de ces garanties.
Même avec un modèle du marché financier très simple, il n'existe pas de formules analytiques fermées et des simulations numériques ou des méthodes d'approximations sont nécessaires. Nous présentons, en plus de l'approche purement numérique, trois méthodes analytiques. La validité de ces trois méthodes sera discutée avec un exemple numérique.

Zusammenfassung

Mit der Einführung der Schweizer Solvenz Tests bzw. der Implementierung von Solvency II müssen die Versicherungsgesellschaften ihren Verträgen einen ökonomischen Wert zuweisen. In diesem Artikel betrachten wir das Beispiel einer fondsgebundenen Versicherung mit periodischen Prämien und Garantien im Falle von Erleben und Tod; für diesen Vertragstyp liegt die Schwierigkeit in der Bewertung dieser Garantien, die, aus finanziellem Gesichtspunkt gesehen, Optionen asiatischen Typs entsprechen. Diese Optionen werden im Markt nicht gehandelt, und die ganze Schwierigkeit liegt in der Auswahl eines geeigneten Modells, das erlaubt, den Wert dieser Garantien zu bestimmen.
Auch mit einem einfachen Kapitalmarktmodell gibt es keine geschlossenen analytischen Formeln, und numerische Simulationen oder Annäherungsverfahren sind notwendig. Wir präsentieren, zusätzlich zur rein numerischen Simulation, drei verschiedene analytische Methoden. Die Gültigkeit dieser drei Methoden wird durch ein numerisches Beispiel aufgezeigt.