A note on mortality selection

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1 Introduction

Although, in general, mortality seems to be decreasing over calendar time, some data materials indicate that mortality has increased for high ages. One explanation could be that because of advances in medicine, persons who in earlier times would have died in low ages, will reach higher ages. These persons could be weaker individuals than those who would have reached this age group in earlier days, so that mortality will increase in this age group. Thus, one could imagine that mortality would decrease in calendar time for low ages, but increase for high ages. On the other hand, one would expect that the decrement function would increase for all ages. In the following, we shall study a simplified setting under which these characteristics are satisfied.

In Section 2, we introduce a cohort from a population of two disjoint groups of people, one with low mortality and one with high mortality, and deduce expressions for the decrement function, mortality force, and expected remaining life-time for this cohort in terms of the mortality force in those two groups.

In Section 3, we assume that for a later cohort from the same population, the mortality force has decreased in the high-risk group, ceteris paribus, and we compare the situation in the aggregate group. Our main result is that although the mortality force has not increased for any age in any of the sub-groups, it might have increased for high ages in the aggregate group.

In Section 4, we argue that because of continuity, it may still have increased for some ages in the aggregate group if it has also decreased moderately in the low-risk group.

Finally, in Section 5, we give a brief verbal comparison with the setting where the mortality force has increased in the low-risk group and is unchanged in the high-risk group.

Vaupel & Yashin (1985) discussed a more general version of the model of Section 2 with a population consisting of two disjoint groups and showed how the mortality in the aggregate group was affected by various assumptions on the mortality in each of these two groups.

They also discussed a situation with two cohorts. The population consists of a low-risk group and a high-risk group. In the second cohort, the mortality force is lower for young ages in both groups than in the first cohort. In the aggregate

group, the mortality force has then become lower for young ages, but higher for older ages.
To avoid messy discussion on regularity, we do not aim at making our assumptions as general as possible.

2 The initial setting

We consider a cohort from a population consisting of two disjoint groups, one with low mortality and one with high mortality, and assume that there is no transition between these groups. The difference between the two groups could originate in e.g. genetic characteristics, including gender.
At age x, the mortality force is μ_x^{L} in the low-risk group and μ_x^{H} in the high-risk group with $\mu_x^{\mathrm{L}} < \mu_x^{\mathrm{H}}$ for all ages. We introduce the corresponding cumulative mortality forces $M_x^{\mathrm{L}} = \int_0^x \mu_y^{\mathrm{L}}\, dy$ and $M_x^{\mathrm{H}} = \int_0^x \mu_y^{\mathrm{H}}\, dy$.
Let the probability that a new-born person belongs to the high-risk group, be $\rho \in (0, 1)$. In that group, the probability that a new-born person will be alive at age x, is $e^{-M_x^{\mathrm{H}}}$, and in the other group, the corresponding probability is $e^{-M_x^{\mathrm{L}}}$. By conditioning on sub-group, we obtain that the probability that a new-born person in the aggregate group will be alive at age x, is

$$l_x = \rho\, e^{-M_x^{\mathrm{H}}} + (1-\rho)\, e^{-M_x^{\mathrm{L}}}\ ; \tag{1}$$

this is the decrement function of the aggregate group.
The mortality force in this group is $\mu_x = -l'_x / l_x$. Insertion of (1) gives

$$\mu_x = \frac{\rho\, \mu_x^{\mathrm{H}}\, e^{-M_x^{\mathrm{H}}} + (1-\rho)\, \mu_x^{\mathrm{L}}\, e^{-M_x^{\mathrm{L}}}}{\rho\, e^{-M_x^{\mathrm{H}}} + (1-\rho)\, e^{-M_x^{\mathrm{L}}}}\,, \tag{2}$$

that is,

$$\mu_x = \rho_x\, \mu_x^{\mathrm{H}} + (1-\rho_x)\, \mu_x^{\mathrm{L}}\,, \tag{3}$$

where

$$\rho_x = \frac{\rho}{\rho + (1-\rho)\, e^{M_x^{\mathrm{H}} - M_x^{\mathrm{L}}}} \tag{4}$$

is the probability that an x year old person is in the high-risk group, that is, the conditional probability that a person belongs to that group given that he is alive at age x. From (4), we see that this probability is decreasing in x. This is intuitively clear. High-risk people will tend to die earlier than low-risk people so the proportion of high-risk people in the aggregate group will decrease with age. In particular, if $\lim_{x\uparrow\infty}\left(M_x^{\mathrm{H}} - M_x^{\mathrm{L}}\right) = \infty$, then $\lim_{x\uparrow\infty}\rho_x = 0$ so that the mortality force in the aggregate group is asymptotically equal to the low-risk mortality force when age goes to infinity. From (4), we see that ρ_x depends on the mortality forces in the two subgroups only through their difference.

At age x, the expected remaining life-time of a person in the aggregate group is

$$\bar{e}_x = \frac{\int_x^\infty l_y\,dy}{l_x} \tag{5a}$$

$$= \frac{\rho\int_x^\infty e^{-M_y^{\mathrm{H}}}\,dy + (1-\rho)\int_x^\infty e^{-M_y^{\mathrm{L}}}\,dy}{\rho\,e^{-M_x^{\mathrm{H}}} + (1-\rho)\,e^{-M_x^{\mathrm{L}}}} \tag{5b}$$

$$= \int_x^\infty e^{-\int_x^y \mu_z\,dz}\,dy\,. \tag{5c}$$

With

$$\bar{e}_x^{\mathrm{H}} = \int_x^\infty e^{M_x^{\mathrm{H}} - M_y^{\mathrm{H}}}\,dy\ ;\quad \bar{e}_x^{\mathrm{L}} = \int_x^\infty e^{M_x^{\mathrm{L}} - M_y^{\mathrm{L}}}\,dy$$

being the expected remaining life-time of a person at age x in the high-risk and low-risk group respectively, we can rewrite (5b) as

$$\bar{e}_x = \rho_x\,\bar{e}_x^{\mathrm{H}} + (1-\rho_x)\,\bar{e}_x^{\mathrm{L}}\,.$$

Analogous to (1), we could also have set up this formula directly by conditioning on sub-group.

Before closing Section 2, let us consider the special case when the mortality forces in the sub-groups are constant, $\mu_x^{\mathrm{H}} = \mu^{\mathrm{H}}$ and $\mu_x^{\mathrm{L}} = \mu^{\mathrm{L}}$ for all x. Then, from the formulae deduced above, we obtain that for all x,

$$l_x = \rho\,e^{-\mu^{\mathrm{H}}x} + (1-\rho)\,e^{-\mu^{\mathrm{L}}x}$$

$$\mu_x = \rho_x\,\mu^{\mathrm{H}} + (1-\rho_x)\,\mu^{\mathrm{L}}\ ;\qquad \rho_x = \frac{\rho}{\rho + (1-\rho)\,e^{(\mu^{\mathrm{H}}-\mu^{\mathrm{L}})x}}$$

$$\bar{e}_x^{\mathrm{H}} = \frac{1}{\mu^{\mathrm{H}}}\ ;\qquad \bar{e}_x^{\mathrm{L}} = \frac{1}{\mu^{\mathrm{L}}}\ ;\qquad \bar{e}_x = \frac{\rho_x}{\mu^{\mathrm{H}}} + \frac{1-\rho_x}{\mu^{\mathrm{L}}}\,.$$

When x goes from zero to infinity, ρ_x decreases from ρ to zero. Hence, μ_x decreases from

$$\mu_0 = \rho\,\mu^{\mathrm{H}} + (1-\rho)\,\mu^{\mathrm{L}}$$

to μ^{L}, and $\bar{e}_x$ increases from

$$\bar{e}_0 = \frac{\rho}{\mu^{\mathrm{H}}} + \frac{1-\rho}{\mu^{\mathrm{L}}}$$

to $1/\mu^{\mathrm{L}}$; the older a person gets, the more likely is it that he belongs to the low-risk group, and, hence, the longer becomes his expected remaining life-time.

3 Decreased mortality force in the high-risk group

Let us now consider a later cohort from the same population and assume that for this cohort, the mortality force in the high-risk group has decreased to $\tilde{\mu}_x^{\mathrm{H}}$ for all ages x, ceteris paribus. We distinguish quantities in the new cohort from those in the old cohort by adding a tilde; we have already introduced the new high-risk mortality force $\tilde{\mu}_x^{\mathrm{H}}$ under this convention.

From (1), we immediately see that

$$\tilde{l}_x > l_x \tag{6}$$

for all x. Furthermore, two applications of (2) give that $\tilde{\mu}_0 < \mu_0$.

If $\tilde{\mu}_x^{\mathrm{H}} \leq \mu_x^{\mathrm{L}}$ for all x, then two applications of (3) give that $\tilde{\mu}_x \leq \mu_x^{\mathrm{L}} < \mu_x$ for all x. From (5c), we then obtain that $\tilde{\bar{e}}_x > \bar{e}_x$ for all x.

The question is now, will we have $\tilde{\mu}_x < \mu_x$ for all x also when $\mu_x^{\mathrm{L}} < \tilde{\mu}_x^{\mathrm{H}} < \mu_x^{\mathrm{H}}$ for all x? Intuitively, one would think so; when the mortality force has decreased in a sub-group, ceteris paribus, then one would believe that it would also have decreased in the aggregate group. On the other hand, for all x, $\tilde{\mu}_x^{\mathrm{H}} < \mu_x^{\mathrm{H}}$, and from (4), we see that that implies that $\tilde{\rho}_x > \rho_x$ for all x, that is, the proportion of high-risk people in the aggregate group will have increased. Could this have increased the mortality force in the aggregate group although the high-risk people will now have lower mortality force than in our original setting? The following theorem addresses this question.

Theorem 1 *If $\mu_x^L < \tilde{\mu}_x^H < \mu_x^H$ for all x and*

$$\lim_{x \uparrow \infty} \left(\tilde{M}_x^H - M_x^L \right) = \infty \tag{7}$$

$$\lim_{x \uparrow \infty} \left(M_x^H - \tilde{M}_x^H \right) = \infty \tag{8}$$

$$\underline{\lim}_{x \uparrow \infty} \frac{\tilde{\mu}_x^H - \mu_x^L}{\mu_x^H - \mu_x^L} > 0, \tag{9}$$

then there exists a positive number y such that $\tilde{\mu}_x > \mu_x$ for all $x > y$.

Proof. For all x, application of (2) and some manipulation gives that

$$\frac{\tilde{\mu}_x - \mu_x^{\mathrm{L}}}{\mu_x - \mu_x^{\mathrm{L}}} = \frac{\rho\, e^{-\left(M_x^{\mathrm{H}} - M_x^{\mathrm{L}}\right)} + 1 - \rho}{\rho\, e^{-\left(\tilde{M}_x^{\mathrm{H}} - M_x^{\mathrm{L}}\right)} + 1 - \rho} \frac{\tilde{\mu}_x^{\mathrm{H}} - \mu_x^{\mathrm{L}}}{\mu_x^{\mathrm{H}} - \mu_x^{\mathrm{L}}}\, e^{M_x^{\mathrm{H}} - \tilde{M}_x^{\mathrm{H}}} . \tag{10}$$

As

$$M_x^{\mathrm{H}} - M_x^{\mathrm{L}} = \left(M_x^{\mathrm{H}} - \tilde{M}_x^{\mathrm{H}} \right) + \left(\tilde{M}_x^{\mathrm{H}} - M_x^{\mathrm{L}} \right) ,$$

(8) and (7) imply that

$$\lim_{x \uparrow \infty} \left(M_x^{\mathrm{H}} - M_x^{\mathrm{L}} \right) = \infty . \tag{11}$$

Together with (7), this gives that the first fraction on the right-hand side of the equality sign in (10) goes to one when x goes to infinity. Combining this with (9) gives that

$$\underline{\lim}_{x \uparrow \infty} \frac{\rho\, e^{-\left(M_x^{\mathrm{H}} - M_x^{\mathrm{L}}\right)} + 1 - \rho}{\rho\, e^{-\left(M_x^{\mathrm{H}} - M_x^{\mathrm{L}}\right)} + 1 - \rho} \frac{\tilde{\mu}_x^{\mathrm{H}} - \mu_x^{\mathrm{L}}}{\mu_x^{\mathrm{H}} - \mu_x^{\mathrm{L}}} > 0 . \tag{12}$$

From (8), we see that $e^{M_x^{\mathrm{H}} - \tilde{M}_x^{\mathrm{H}}}$ goes to infinity when x goes to infinity. Together with (10) and (12), this gives that

$$\lim_{x \uparrow \infty} \frac{\tilde{\mu}_x - \mu_x^{\mathrm{L}}}{\mu_x - \mu_x^{\mathrm{L}}} = \infty .$$

Then there must exist a positive number y such that

$$\frac{\tilde{\mu}_x - \mu_x^{\mathrm{L}}}{\mu_x - \mu_x^{\mathrm{L}}} > 1$$

for all $x > y$, so that $\tilde{\mu}_x > \mu_x$ for all $x > y$. This proves the theorem. Q.E.D.

Remarks :

1. The conditions of Theorem 1 are independent of ρ. However, the value of y will depend on ρ.

2. From (11), (7), and (4), we see that in both cohorts, the mortality force in the aggregate group approaches the mortality force in the low-risk group when age goes to infinity.

3. From (5a) and (6) follows that $\tilde{\bar{e}}_0 > \bar{e}_0$ whereas (5c) gives that $\tilde{\bar{e}}_x < \bar{e}_x$ for all $x > y$.

4. By interchanging the two cohorts, we see that if the high-risk mortality force has increased for all ages, ceteris paribus, then, under the accordingly adapted assumptions of Theorem 1, the mortality force in the aggregate group will have decreased for high ages.

Let us now consider the special case when there exist constants α and $\tilde{\alpha}$ with $\alpha > \tilde{\alpha} > 0$ such that $\mu_x^{\mathrm{H}} = \alpha + \mu_x^{\mathrm{L}}$ and $\tilde{\mu}_x^{\mathrm{H}} = \tilde{\alpha} + \mu_x^{\mathrm{L}}$ for all x. Insertion in (3) and (4) gives

$$\mu_x = \mu_x^{\mathrm{L}} + \rho_x \, \alpha \, ; \qquad \tilde{\mu}_x = \mu_x^{\mathrm{L}} + \tilde{\rho}_x \, \tilde{\alpha} \tag{13}$$

$$\rho_x = \frac{\rho}{\rho + (1-\rho)\, e^{\alpha x}} \, ; \qquad \tilde{\rho}_x = \frac{\rho}{\rho + (1-\rho)\, e^{\tilde{\alpha} x}} \, . \tag{14}$$

We easily see that the conditions of Theorem 1 are satisfied in this case, and (10) reduces to

$$\frac{\tilde{\mu}_x - \mu_x^{\mathrm{L}}}{\mu_x - \mu_x^{\mathrm{L}}} = \frac{\tilde{\alpha}}{\alpha} \frac{\rho + (1-\rho)\, e^{\alpha x}}{\rho + (1-\rho)\, e^{\tilde{\alpha} x}} \, . \tag{15}$$

This gives

$$\frac{d}{dx} \frac{\tilde{\mu}_x - \mu_x^{\mathrm{L}}}{\mu_x - \mu_x^{\mathrm{L}}} =$$
$$\frac{\tilde{\alpha}}{\alpha} \frac{1-\rho}{\left(\rho + (1-\rho)\, e^{\tilde{\alpha} x}\right)^2} \left(\rho \left(\alpha e^{\alpha x} - \tilde{\alpha} e^{\tilde{\alpha} x}\right) + (1-\rho)\,(\alpha - \tilde{\alpha})\, e^{(\alpha + \tilde{\alpha})x}\right) > 0 \, ,$$

that is, $\left(\tilde{\mu}_x - \mu_x^{\mathrm{L}}\right) / \left(\mu_x - \mu_x^{\mathrm{L}}\right)$ is strictly increasing in x. Thus, there exists a unique positive number y such that $\tilde{\mu}_y = \mu_y$, $\tilde{\mu}_x < \mu_x$ when $0 \leq x < y$, and $\tilde{\mu}_x > \mu_x$ when $x > y$. From (15), we obtain that this y is determined by

$$\frac{\tilde{\alpha}}{\alpha} \frac{\rho + (1-\rho)\, e^{\alpha y}}{\rho + (1-\rho)\, e^{\tilde{\alpha} y}} = 1 \, ,$$

that is,

$$\tilde{\alpha}\, e^{\alpha y} - \alpha\, e^{\tilde{\alpha} y} = \frac{\rho}{1-\rho}\,(\alpha - \tilde{\alpha})\,. \qquad (16)$$

For a numerical example, we let $\rho = 0.9$, $\alpha = 0.1$, $\tilde{\alpha} = 0.06$, and $\mu_x^{\mathrm{L}} = 0.1$ for all x. The remaining life-time will then converge to $1/\mu^{\mathrm{L}} = 10$ when age goes to infinity. By solving (16) numerically, we obtain that $y = 26.52$. The mortality force, decrement function, and expected remaining life-time in the aggregate group in the two cohorts are displayed in Figures 1–3. The diagrams confirm the deductions presented above.

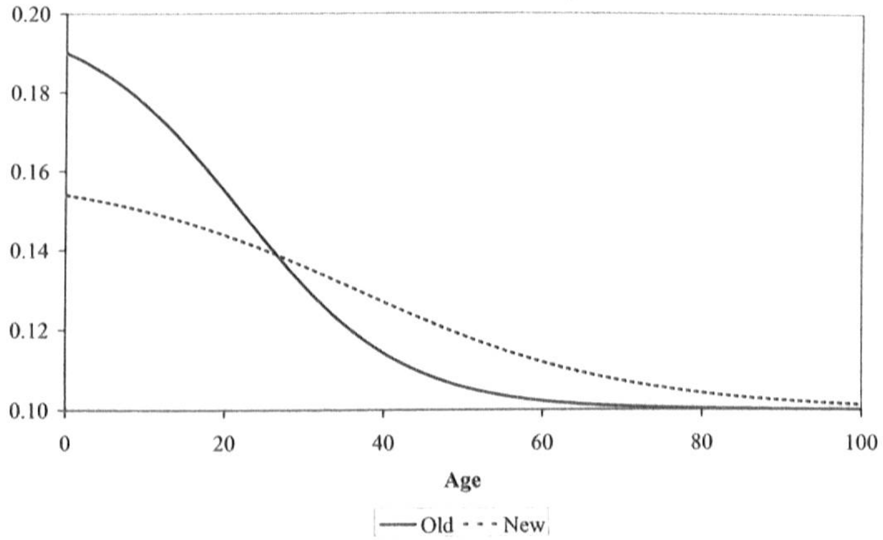

Figure 1: Mortality force in the two cohorts.

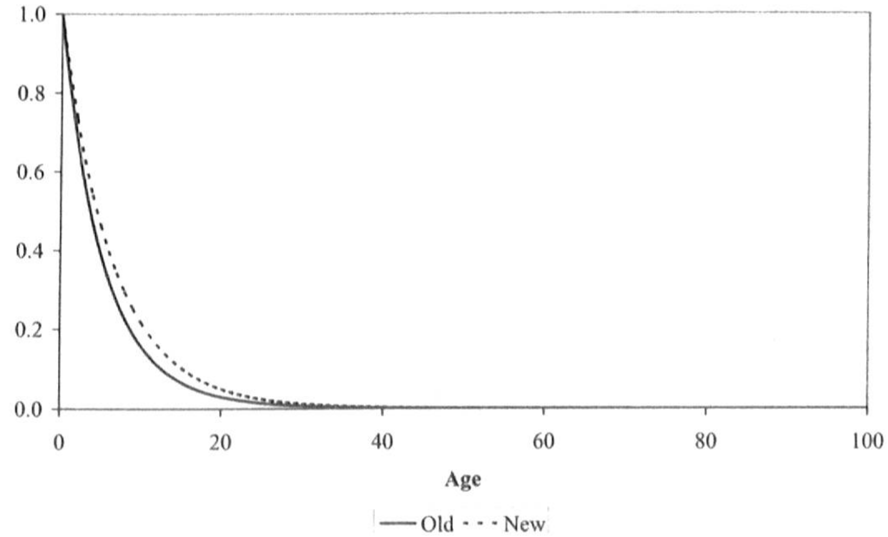

Figure 2: Decrement function in the two cohorts.

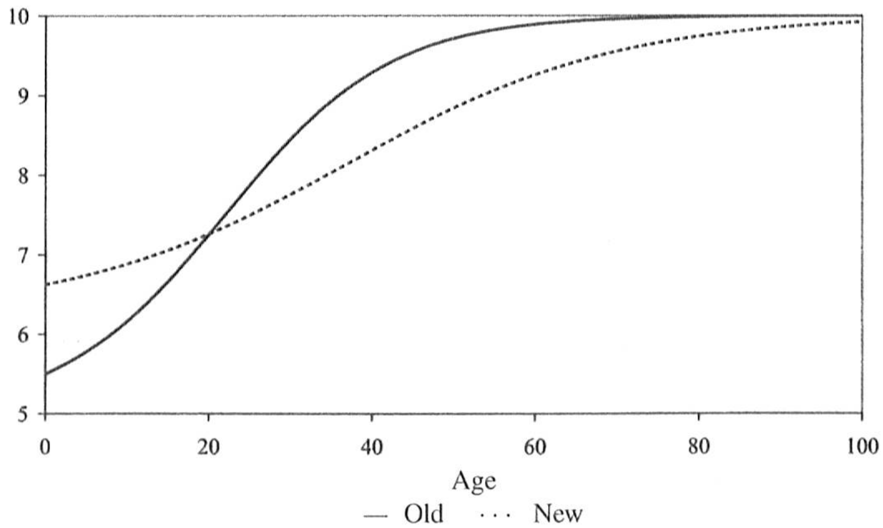

Figure 3: Expected remaining life-time in the two cohorts.

4 Decreased mortality force in both groups

It is tempting to modify the setting of Section 3 by assuming that the mortality force has decreased in the low-risk group too.

By a continuity argument, it is clear that in this setting, we can still have higher mortality in high ages in the aggregate group than in our original cohort : Let $\tilde{\mu}_{x,\varepsilon}$ denote the mortality force in the aggregate group if we change the mortality force in the low-risk group to $\mu^{\mathrm{L}}_{x,\varepsilon} = \mu^{\mathrm{L}}_x - \varepsilon$ and let it still be $\tilde{\mu}^{\mathrm{H}}_x$ in the high-risk group. In particular, we have $\tilde{\mu}_{x,0} = \tilde{\mu}_x$. From (2), we obtain that

$$\tilde{\mu}_{x,\varepsilon} = \frac{\rho\, \tilde{\mu}^{\mathrm{H}}_x\, e^{-\tilde{M}^{\mathrm{H}}_x} + (1-\rho)\left(\mu^{\mathrm{L}}_x - \varepsilon\right) e^{-M^{\mathrm{L}}_x + \varepsilon x}}{\rho\, e^{-\tilde{M}^{\mathrm{H}}_x} + (1-\rho)\, e^{-M^{\mathrm{L}}_x + \varepsilon x}} \,.$$

If $\tilde{\mu}_x > \mu_x$ for some x, then we also have $\tilde{\mu}_{x,\varepsilon} > \mu_x$ for $\varepsilon > 0$ sufficiently small as $\tilde{\mu}_{x,\varepsilon}$ is continuous in ε for all values of x.

If, for all ages x, μ^{H}_x decreases to $\tilde{\mu}^{\mathrm{H}}_x$ like in Section 3 and also μ^{L}_x decreases to $\tilde{\mu}^{\mathrm{L}}_x$, then the natural extension of (10) seems to be

$$\frac{\tilde{\mu}_x - \tilde{\mu}^{\mathrm{L}}_x}{\mu_x - \mu^{\mathrm{L}}_x} = \frac{\rho\, e^{-\left(M^{\mathrm{H}}_x - M^{\mathrm{L}}_x\right)} + 1 - \rho}{\rho\, e^{-\left(\tilde{M}^{\mathrm{H}}_x - \tilde{M}^{\mathrm{L}}_x\right)} + 1 - \rho}\, \frac{\tilde{\mu}^{\mathrm{H}}_x - \tilde{\mu}^{\mathrm{L}}_x}{\mu^{\mathrm{H}}_x - \mu^{\mathrm{L}}_x}\, e^{\left(M^{\mathrm{H}}_x - M^{\mathrm{L}}_x\right) - \left(\tilde{M}^{\mathrm{H}}_x - \tilde{M}^{\mathrm{L}}_x\right)} \,.$$

It is trivial to extend the regularity conditions of Theorem 1 to obtain

$$\lim_{x \uparrow \infty} \frac{\tilde{\mu}_x - \tilde{\mu}^{\mathrm{L}}_x}{\mu_x - \mu^{\mathrm{L}}_x} = \infty \,.$$

Then there must exist a positive number y such that

$$\frac{\tilde{\mu}_x - \tilde{\mu}^{\mathrm{L}}_x}{\mu_x - \mu^{\mathrm{L}}_x} > 1$$

for all $x > y$. This implies that $\tilde{\mu}_x > \mu_x - \left(\mu^{\mathrm{L}}_x - \tilde{\mu}^{\mathrm{L}}_x\right)$ for all $x > y$. However, as $\mu^{\mathrm{L}}_x > \tilde{\mu}^{\mathrm{L}}_x$, this does not tell us whether $\tilde{\mu}_x > \mu_x$.

5 Increased mortality force in the low-risk group

In Section 3, we discussed the effect of a decrease of the high-risk mortality force for all ages, ceteris paribus. We showed that although the mortality force had not increased for any age in any of the two sub-groups, it could have increased for high ages in the aggregate group. It is tempting to ask whether we could have a similar apparent paradox with an increase of the mortality force in the low-risk group and unchanged mortality force in the high-risk group; could that give a decrease in the mortality force in the aggregate group for high ages? The answer is no. In the setting of Section 3, a decrease in the high-risk mortality force for all ages implied that a larger proportion of the people in the aggregate group would be in the high-risk group, and under the assumptions of Theorem 1, for high ages this effect would increase the mortality force in the aggregate group, although the high-risk people would now have a lower mortality than earlier. How would that be in our present setting? The increase in the low-risk mortality force implies a decrease in the proportion of low-risk people in the aggregate group. That in itself increases the mortality force in the aggregate group. In addition, the low-risk people will now have a higher mortality force, and that will further increase the mortality force in the aggregate group. Hence, that mortality force will be increased for all ages. Whereas in the setting of Section 3, the decreased high-risk mortality force had two effects to the aggregate group that worked in converse directions, in our present setting, the analogous effects to the mortality force in the aggregate group from the increased low-risk mortality force work in the same direction.

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Abstract

In general, mortality usually seems to decrease over calender time. However, some data materials indicate an increase for higher ages. In this note, we present a simple setting in which such an effect can be explained.

Zusammenfassung

Im Allgemeinen erscheint es, als ob die Sterblichkeit mit der Kalenderzeit abnimmt. Dennoch geben einige Datenmateriale den Eindruck, dass die Sterblichkeit in den hohen Altersstufen zunimmt. In dieser Notiz präsentieren wir ein Modell, mit welchem ein solcher Effekt erklärt werden kann.

Résumé

En général, la mortalité semble décroître avec le temps calendrier. Cependant, quelques données matérielles indiquent une augmentation pour les âges avancés. Dans cette note, on présente un cadre simple qui permet d'expliquer cet effet.